APPLIED DIFFERENTIAL CALCULUS
LECTURE 7: Fourier series and separation of variables: Laplace equation.
PROBLEMS

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Problem 1

Find the solution of the following boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad x\in(0,1)\,,\quad y\in(0,1)\\
&u(0,y)=0\,,\ u(1,y)=0\,,\quad y\in[0,1],\\
&\frac{\partial u}{\partial y}(x,0)=1\,,\ u(x,1)=0,\quad x\in[0,1]\,.
\end{aligned}$$

SOLUTION:
Separation of variables, $u = X(x)Y(y)$ yields $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$. The homogeneous boundary conditions produce $X(0) = 0$, $X(1) = 0$, $Y(1) = 0$ and the only inhomogeneous boundary condition occurs at $y = 0$. Thus we assume that $\lambda > 0$ and solve the eigenvalue problem: $X'' + \lambda X = 0$, $X(0) = 0$, $X(1) = 0$. The solution is $\lambda_n = n^2\pi^2$, $X_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$. Why? The general solution of the ODE is $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. $X(0) = c_1 = 0$ and $X(1) = c_2 \sin(\sqrt{\lambda}) = 0$, which yields $\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. Solving now the problem $Y'' - \lambda Y = 0$ with $Y(1) = 0$, we obtain $X_n(x) = \sinh[\sqrt{\lambda_n}(y-1)] = \sinh[n\pi(y-1)]$. By superposition:

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin(n\pi x)\sinh[n\pi(y-1)],$$

$$1=\sum_{n=1}^{\infty} a_n n\pi \sin(n\pi x)\cosh(-n\pi)=\sum_{n=1}^{\infty} a_n n\pi\cosh(n\pi)\sin(n\pi x).$$

Then

$$n\pi\cosh(n\pi)a_n = 2\int_0^1 \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)|_0^1 = 2\frac{1-\cos n\pi}{n\pi}.$$

Thus a_n is

$$a_n = \begin{cases} \frac{4}{n^2\pi^2\cosh(n\pi)}, & \text{odd } n, \\ 0, & \text{even } n. \end{cases}$$

and the solution of the BVP is

$$u(x,t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi x]\sinh[(2n-1)\pi(y-1)]}{(2n-1)^2\cosh[(2n-1)\pi]}.$$

Problem 2 Find the solution of the following boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x \in (0,1)\,, \quad y \in (0,1)$$
$$u(0,y) = 0\,, \; u(1,y) = \sin\frac{\pi y}{2}\,, \quad y \in [0,1],$$
$$\frac{\partial u}{\partial y}(x,0) = 0\,, \; \frac{\partial u}{\partial y}(x,1) = 0, \quad x \in [0,1]\,.$$

SOLUTION:
Separation of variables, $u = X(x)Y(y)$ yields $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$. The homogeneous boundary conditions produce $X(0) = 0$, $Y'(0) = 0$, $Y'(1) = 0$ and the only inhomogeneous boundary condition occurs at $x = 1$. Thus we assume that $\lambda > 0$ and solve the eigenvalue problem: $Y'' + \lambda Y = 0$, $Y'(0) = 0$, $Y'(1) = 0$. The solution is $\lambda_n = n^2\pi^2$, $Y_n(y) = \cos(n\pi y)$, $n = 0, 1, 2, \ldots$. Why? The general solution of the ODE is $Y = c_1\cos(\sqrt{\lambda}y) + c_2\sin(\sqrt{\lambda}y)$. $Y'(0) = \sqrt{\lambda}c_2 = 0$ and $Y'(1) = \sqrt{\lambda}c_1\sin(\sqrt{\lambda}) = 0$, which yields $c_2 = 0$, $\sqrt{\lambda} = n\pi$, $n = 0, 1, 2, \ldots$. Solving now the problem $X'' - \lambda X = 0$ with $X(0) = 0$, we obtain $X_n(x) = \sinh[\sqrt{\lambda_n}x] = \sinh(n\pi x)$. For $n = 0$, $X_0 = a_0 x$. By superposition:

$$u(x,y) = a_0 x + \sum_{n=1}^{\infty} a_n \sinh(n\pi x)\cos(n\pi y),$$

$$\sin\frac{\pi y}{2} = a_0 + \sum_{n=1}^{\infty} a_n \sinh(n\pi)\cos(n\pi y).$$

Then

$$a_0 = \int_0^1 \sin\frac{\pi y}{2}dy = \frac{2}{\pi},$$
$$a_n = 2\int_0^1 \sin\frac{\pi y}{2}\cos(n\pi y)dy = \int_0^1\left(\sin\frac{(2n+1)\pi y}{2} - \sin\frac{(2n-1)\pi y}{2}\right)dy$$
$$= \frac{2}{(2n+1)\pi} - \frac{2}{(2n-1)\pi} = -\frac{4}{(4n^2-1)\pi}.$$

Thus the solution of the BVP is

$$u(x,y) = \frac{2x}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\sinh(n\pi x)\cos(n\pi y).$$

Problem 3 Find the solution of the following boundary value problem:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x \in (0,1)\,, \quad y \in (0,1) \\
&u(0,y) = 0\,, \; u(1,y) = 0\,, \; y \in [0,1], \\
&\frac{\partial u}{\partial y}(x,0) = e^{2x}\,, \; u(x,1) = 0, \; x \in [0,1]\,.
\end{aligned}$$

SOLUTION:

Separation of variables, $u = X(x)Y(y)$ yields $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$. The homogeneous boundary conditions produce $X(0) = 0$, $X(1) = 0$, $Y(1) = 0$ and the only non-homogeneous boundary condition occurs at $y = 0$. Thus we assume that $\lambda > 0$ and solve the eigenvalue problem: $X'' + \lambda X = 0$, $X(0) = 0$, $X(1) = 0$. The solution is $\lambda_n = n^2\pi^2$, $X_n(x) = \sin(n\pi x)$, $n = 1, 2, \ldots$. Why? The general solution of the ODE is $X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$. $X(0) = c_1 = 0$ and $X(1) = c_2 \sin(\sqrt{\lambda}) = 0$, which yields $\sqrt{\lambda} = n\pi$, $n = 1, 2, \ldots$. Solving now the problem $Y'' - \lambda Y = 0$ with $Y(1) = 0$, we obtain $X_n(x) = \sinh[\sqrt{\lambda_n}(y-1)] = \sinh[n\pi(y-1)]$. By superposition:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\sinh[n\pi(y-1)],$$

$$e^{2x} = \sum_{n=1}^{\infty} a_n n\pi \sin(n\pi x)\cosh(-n\pi) = \sum_{n=1}^{\infty} a_n n\pi \cosh(n\pi)\sin(n\pi x).$$

Then

$$\begin{aligned}
n\pi\cosh(n\pi)a_n = 2\int_0^1 e^{2x}\sin(n\pi x)dx = 2\mathrm{Im}\int_0^1 e^{2x+in\pi x}dx = \mathrm{Im}\frac{2}{2+in\pi}(e^{2+in\pi} - 1) \\
= \mathrm{Im}\frac{2}{2+in\pi}[e^2(-1)^n - 1] = -\frac{2n\pi}{4+n^2\pi^2}[(-1)^n e^2 - 1].
\end{aligned}$$

Thus a_n is

$$a_n = \frac{2[1-(-1)^n e^2]}{(4+n^2\pi^2)\cosh(n\pi)},$$

and the solution of the BVP is

$$u(x,t) = 2\sum_{n=1}^{\infty} \frac{1-(-1)^n e^2}{(4+n^2\pi^2)\cosh(n\pi)} \sin(n\pi x)\sinh[n\pi(y-1)].$$
