Cálculo Diferencial Aplicado

FINAL EXAM 1

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Problem 1 (2.0 mark) Consider the following ordinary differential equation (ODE)

$$y'' + y = xe^{x} + 2e^{-x}.$$

i) Find the general solution of the ODE.

ii) Find the solution satisfying the initial conditions $y(0) = 1\,,\ y'(0) = 1\,.$

Problem 2 (2.0 mark) Consider the following system of first-order differential equations

$$\vec{X}'(t) = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \vec{X}(t)$$

for $t > 0\,.$

i) Find the general solution of the system and verify the results.

ii) Analyze the behavior of the solution calculated in (i) as $t \to +\infty\,.$ Can this behavior depend on the initial condition possibly assigned to the system?

Problem 3 (2.0 mark) Let $g(x) = -1 - x/2\,.$ Then, solve the following initial value problem

$$\begin{cases} -\dfrac{2y + x}{y + x}\, y' = \dfrac{2y}{x^2}\,(1 + g)\,, & x \geq 1\,, \\ y(1) = 1\,. \end{cases}$$

Problem 4 (2.0 mark) Consider the following heat equation

$$\begin{aligned}
\text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)\,, \quad t>0\,, \quad x \in (0,\pi) \\
\text{Boundary Conditions (BCs)} &: \quad u(0,t) = 0\,, \quad u(\pi,t) = 0\,, \quad t>0 \\
\text{Initial Condition (IC)} &: \quad u(x,0) = f(x)\,, \quad x \in [0,\pi]\,.
\end{aligned}$$

Apply a separation of variables as $u(x,t) = X(x)\,T(t) \not\equiv 0$.

i) Prove that $T(t) = ce^{-\lambda t}$, where $c \in \mathbb{R} \setminus \{0\}$ and λ is the separation constant.

ii) Prove that $X(x)$ satisfies the following boundary value problem

$$X'' + \lambda X = 0\,, \quad X(0) = 0\,, \, X(\pi) = 0\,,$$

and find the values of $\lambda > 0$ providing nonzero solutions.

iii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} A_n\, e^{-n^2 t} \sin(nx)\,, \quad \text{with } A_n \in \mathbb{R}\,,$$

find the coefficients A_n $(n \in \mathbb{N})$ supposing that $f(x) = \sin^3(x)$.

Hint. The following results can be useful.

- Given $L > 0$ and $m\,, n \in \mathbb{N}$, we have $\displaystyle\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x = \begin{cases} 0\,, & m \neq n \\ L/2\,, & m = n\,. \end{cases}$
- $\sin(3x) = \sin(x)\left(3 - 4\sin^2(x)\right)\,, \ \forall x \in \mathbb{R}\,.$

Problem 5 (2.0 mark) Consider the initial value problem

$$\begin{cases} y' &= 1 + \dfrac{y}{2} \\ y(0) &= 0 \end{cases}$$

to which the following numerical scheme (*improved Euler*) is applied

$$Y_{n+1} = Y_n + \frac{h}{2}\left[\, f(t_n, Y_n) + f(t_{n+1}, Y_n + hf(t_n, Y_n))\,\right].$$

i) Calculate, by using steps $h_1 = 0.2$ and $h_2 = 0.1$, the quantities $Y^{h_1}_{t=0.4}$ and $Y^{h_2}_{t=0.4}$, which are approximated values of $y(0.4)$.

ii) Estimate the order of the numerical method by means of the results obtained in (i), noting that the exact solution of the problem is $y(t) = 2\,(e^{t/2} - 1)$.
