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Cálculo Diferencial Aplicado

FINAL EXAM 2

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Problem 1 (2.0 mark) Consider the differential equation $x^2y'' - 3xy' + 4y = \ln x\,, \quad x > 0\,.$

(i) Apply a change of variable to transform it into an equation with constant coefficients.

(ii) Solve the obtained equation together with the conditions $y(1) = 1/2\,, \quad y'(1) = 1\,.$

Problem 2 (2.0 mark) Consider the following system of first-order differential equations

$$\begin{pmatrix} X_1'(t) \\ X_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

with $\alpha \in \mathbb{R}$ and $t > 0\,.$

(i) Find the value of α for which the qualitative behavior of the solutions of the system changes (*hint*: calculate the eigenvalues of the coefficients matrix in terms of α). Justify your answer.

(ii) Find the solution of the system for $\alpha = 1$ and $(X_1(0), X_2(0)) = (1, 0)\,.$ In addition, calculate the distance $d(t)$ between the position $(0, 0)$ and the position of a particle that moves according to the computed solution (*hint*: use formula $d(t) = \sqrt{X_1(t)^2 + X_2(t)^2}\,$).

Problem 3 (2.0 mark) Solve the following initial value problem

$$\begin{cases} x^2 + e^{\,y} + (xe^{\,y} + \cos y)\, y' &= 0 \\ y(0) &= g(\pi/2)\,, \end{cases}$$

knowing that the function g verifies

$$g'(x) = \sin(x)\,, \quad g(0) = -1\,.$$

Problem 4 (2.0 mark) Consider the following model for the heat equation.

$$\begin{aligned} \text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)\,, \;\; t>0\,, \;\; x \in (0,\pi/3) \\ \text{Boundary Conditions (BC)} &: \quad \frac{\partial u}{\partial x}(0,t) = 0\,, \;\; \frac{\partial u}{\partial x}(\pi/3,t) = 0\,, \;\; t>0 \\ \text{Initial Condition (IC)} &: \quad u(x,0) = f(x)\,, \;\; x \in [0,\pi/3]\,. \end{aligned}$$

Apply the separation of variables $u(x,t) = X(x)\,T(t) \not\equiv 0$.

(i) Prove that $X(x)$ satisfies the following boundary value problem

$$X'' + \lambda X = 0\,, \quad X'(0) = 0\,, \quad X'(\pi/3) = 0\,,$$

and find the values of the separation constant $\lambda \geq 0$ providing nonzero solutions.

(ii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-9n^2 t} \cos(3nx)\,, \quad \text{with } A_n \in \mathbb{R}\,,$$

find the approximate value of $u(\pi/6, 1/9)$ by considering the first three terms only of the previous series and taking $f(x) = 2x+1$.

Note. The following result may be useful.

Given $L>0$ and $m\,, n \in \mathbb{N} \cup \{0\}$, we have $\displaystyle\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 0\,; & m \neq n \\ L/2\,; & m = n \neq 0 \\ L\,; & m = n = 0\,. \end{cases}$

Problem 5 (2.0 mark) The following initial value problem

$$\begin{cases} y' &= t + \dfrac{y}{2} + 1 \\ y(0) &= 1 \end{cases}$$

must be numerically solved by using the Adams–Bashforth scheme

$$Y_{n+2} = Y_{n+1} + \frac{3}{2}\,h\,f(t_{n+1}, Y_{n+1}) - \frac{1}{2}\,h\,f(t_n, Y_n)\,.$$

(i) Calculate the approximated solution $Y^{h_1}_{t=0.3}$ of $y(0.3)$ with step $h_1 = 0.1$, knowing that Y_1 has to be computed by the explicit Euler method.

(ii) After noting that step $h_2 = 0.01$ yields the approximation $Y^{h_2}_{t=0.3} = 1.5327258$, estimate the order of the numerical scheme by means of $Y^{h_1}_{t=0.3}$, $Y^{h_2}_{t=0.3}$, and the exact solution given by $y(t) = 7e^{t/2} - 2(t+3)$.
