Cálculo Diferencial Aplicado

FINAL EXAM 1

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Problem 1 (2.0 mark) Consider the following ordinary differential equation (ODE)

$$y'' + y = xe^{x} + 2e^{-x}.$$

i) Find the general solution of the ODE.

ii) Find the solution satisfying the initial conditions $y(0) = 1$, $y'(0) = 1$.

SOLUCIÓN:

i) Let us first solve the associated homogeneous equation. The roots of the characteristic equation $r^2 + 1 = 0$ are $\pm i$, thus the general solution of the homogeneous equation is given by $y_h(x) = c_1 \cos(x) + c_2 \sin(x)$, where c_1 and c_2 are two arbitrary constants.
Now, a particular solution of the given nonhomogeneous equation can be found by the method of undetermined coefficients (together with the principle of superposition) in the form $y_p(x) = (Ax + B)e^{x} + Ce^{-x}$. Hence, we get $A = 1/2$, $B = -1/2$, and $C = 1$.
Finally, the general solution of the ODE is

$$\boxed{y(x) = y_h(x) + y_p(x) = c_1 \cos(x) + c_2 \sin(x) + \tfrac{1}{2}(x-1)e^{x} + e^{-x}.}$$

ii) After imposing the given initial conditions, we get $c_1 = 1/2$ and $c_2 = 2$. Thus, the desired solution is

$$\boxed{y(x) = \tfrac{1}{2}\cos(x) + 2\sin(x) + \tfrac{1}{2}(x-1)e^{x} + e^{-x}.}$$

Problem 2 (2.0 mark) Consider the following system of first-order differential equations

$$\vec{X}'(t) = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \vec{X}(t)$$

for $t > 0$.

i) Find the general solution of the system and verify the results.

ii) Analyze the behavior of the solution calculated in (i) as $t \to +\infty$. Can this behavior depend on the initial condition possibly assigned to the system?

SOLUCIÓN:

i) Let us solve the system by calculating the eigenvalues λ of the coefficients matrix, which are both real and equal to $\lambda = -2$. One associated eigenvector is then $\vec{u} = (1, 2)^T$, where the symbol ^{T} stands for the transpose. Thus, the general solution of the system is given by

$$\vec{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{-2t} \right],$$

where c_1, c_2 are two arbitrary constants and the vector $\vec{w} = (w_1, w_2)^T$ satisfies the generalized eigenvalue problem

$$\begin{pmatrix} 2-\lambda & -2 \\ 8 & -6-\lambda \end{pmatrix} \vec{w} = \vec{u} ,$$

with λ and $\vec{u}$ as given above. Thus, it follows that $\vec{w} = (1/2, 1/2)^T$. Finally, upon calculation of the involved derivatives, the validity of the results can be readily checked.

ii) Taking into account that

$$\lim_{t\to+\infty} e^{-2t} = 0 , \qquad \lim_{t\to+\infty} te^{-2t} = 0 ,$$

we can conclude that, independently of the values of c_1 and c_2, the behavior of the general solution of the system as $t \to +\infty$ is given by

$$\lim_{t\to+\infty} \vec{X}(t) = \lim_{t\to+\infty} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Since a possible initial condition only affects the values of c_1 and c_2, then such condition will not influence the system's behavior as $t \to +\infty$.

Problem 3 (2.0 mark) Let $g(x) = -1 - x/2$. Then, solve the following initial value problem

$$\begin{cases} -\dfrac{2y+x}{y+x}\,y' &= \dfrac{2y}{x^2}\,(1+g)\,, \quad x \geq 1\,, \\ y(1) &= 1\,. \end{cases}$$

SOLUCIÓN:
After substituting $g(x)$ into the given differential equation, we get

$$-\frac{2y+x}{y+x}\,y' = \frac{2y}{x^2}\left(-\frac{x}{2}\right) \quad \Longrightarrow \quad \frac{-2y-x}{y+x}\,y' = -\frac{y}{x}\,.$$

This is a nonlinear, homogeneous, first-order ODE. In order to solve it, we can use the change of variable $v = y/x$, which yields $xv' + v = y'$ and

$$\frac{-2y-x}{y+x}\,y' = -\frac{y}{x} \quad \Longrightarrow \quad \frac{-\dfrac{2y}{x}-1}{\dfrac{y}{x}+1}\,y' = -\frac{y}{x}$$

$$\frac{-2v-1}{v+1}\,(xv'+v) = -v \quad \Longrightarrow \quad xv' = -\frac{v^2}{2v+1} \quad \Longrightarrow \quad -\frac{2v+1}{v^2}\,v' = \frac{1}{x}\,.$$

Then, solving the last separable equation, we get

$$-2\ln|v| + \frac{1}{v} - \ln x = c\,,$$

where c is an arbitrary constant. Finally, using $v = y/x$ and the initial condition $y(1) = 1$, an implicit expression for the desired solution is obtained as

$$\boxed{-2\ln\left|\frac{y}{x}\right| + \frac{x}{y} - \ln x = 1\,.}$$

Problem 4 (2.0 mark) Consider the following heat equation

$$\begin{aligned} \text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)\,, \quad t > 0\,, \quad x \in (0,\pi) \\ \text{Boundary Conditions (BCs)} &: \quad u(0,t) = 0\,, \quad u(\pi,t) = 0\,, \quad t > 0 \\ \text{Initial Condition (IC)} &: \quad u(x,0) = f(x)\,, \quad x \in [0,\pi]\,. \end{aligned}$$

Apply a separation of variables as $u(x,t) = X(x)\,T(t) \not\equiv 0$.

i) Prove that $T(t) = ce^{-\lambda t}$, where $c \in \mathbb{R} \setminus \{0\}$ and λ is the separation constant.

ii) Prove that $X(x)$ satisfies the following boundary value problem

$$X'' + \lambda X = 0\,, \quad X(0) = 0\,,\; X(\pi) = 0\,,$$

and find the values of $\lambda > 0$ providing nonzero solutions.

iii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \, e^{-n^2 t} \sin(nx) \,, \quad \text{with } A_n \in \mathbb{R} \,,$$

find the coefficients A_n ($n \in \mathbb{N}$) supposing that $f(x) = \sin^3(x)$.

Hint. The following results can be useful.

- Given $L > 0$ and $m\,, n \in \mathbb{N}$, we have $\displaystyle\int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x = \begin{cases} 0\,, & m \neq n \\ L/2\,, & m = n\,. \end{cases}$
- $\sin(3x) = \sin(x)\left(3 - 4\sin^2(x)\right)\,, \ \forall x \in \mathbb{R}\,.$

SOLUCIÓN:

i) After applying the separation of variables to the PDE, we get

$$X''T = XT' \quad \Longrightarrow \quad \frac{T'}{T} = \frac{X''}{X} = -\lambda\,,$$

where λ is the separation constant. Thus, the first equality $T'/T = -\lambda$ yields the first-order, linear ordinary differential equation $T' + \lambda T = 0$, whose nonzero solutions are given by $T(t) = ce^{-\lambda t}$, where $c \in \mathbb{R} \setminus \{0\}$.

ii) The second equality $X''/X = -\lambda$ above provides the second-order, linear ordinary differential equation $X'' + \lambda X = 0$. In order to obtain the values of X on the boundaries, we need to apply the BCs as

$$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0\,; \quad \text{hence the equality holds } \forall t > 0\,,\ T(t) \not\equiv 0\,;$$

$$u(\pi,t) = X(\pi)T(t) = 0 \implies X(\pi) = 0\,; \quad \text{hence the equality holds } \forall t > 0\,,\ T(t) \not\equiv 0\,.$$

Now, let $\lambda = a^2 > 0$, with $a > 0$. Then, the characteristic equation is given by $r^2 + a^2 = 0$, which yields $r = \pm ia\,,\ i \in \mathbb{C}$. Hence

$$X(x) = c_1 \cos(ax) + c_2 \sin(ax)\,,$$

where c_1 and c_2 are two arbitrary constants. Finally, $X(0) = 0 \implies c_1 = 0$, while $X(\pi) = 0 \implies c_2 \sin(a\pi) = 0$, which ($c_2 \neq 0$) provides $\sin(a\pi) = 0 \implies a = n\,,\ n = 1, 2, 3, \ldots$. Thus, we get $\lambda = n^2\,,\ n = 1, 2, 3, \ldots$.

iii) The given IC yields

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x) = \sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\,,$$

where the second hint above has been used. Then, by matching terms in the series with terms in the right-hand side of this equality, we conclude that

$$\boxed{A_1 = \tfrac{3}{4}\,, \qquad A_2 = 0\,, \qquad A_3 = -\tfrac{1}{4}\,, \qquad A_n = 0 \ \forall n \geq 4\,.}$$

Another (much more involved) way of solving the same task consists in calculating the coefficients by means of the formula $A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\mathrm{d}x$, which is obtained using the first hint above with $L = \pi$.

Problem 5 (2.0 mark) Consider the initial value problem

$$\begin{cases} y' &= 1 + \dfrac{y}{2} \\ y(0) &= 0 \end{cases}$$

to which the following numerical scheme (*improved Euler*) is applied

$$Y_{n+1} = Y_n + \frac{h}{2}\left[\, f(t_n, Y_n) + f(t_{n+1}, Y_n + hf(t_n, Y_n))\,\right].$$

i) Calculate, by using steps $h_1 = 0.2$ and $h_2 = 0.1$, the quantities $Y^{h_1}_{t=0.4}$ and $Y^{h_2}_{t=0.4}$, which are approximated values of $y(0.4)$.

ii) Estimate the order of the numerical method by means of the results obtained in (i), noting that the exact solution of the problem is $y(t) = 2\,(e^{\,t/2} - 1)$.

SOLUCIÓN:

i) By applying the given numerical scheme to the initial value problem we get for each $n = 0, 1, 2, \ldots$

$$Y_{n+1} = Y_n + \frac{h}{2}\left[\, 1 + \frac{Y_n}{2} + 1 + \frac{1}{2}\left(Y_n + h\left(1 + \frac{Y_n}{2}\right)\right)\right]$$

$$Y_{n+1} = Y_n + h\left(1 + \frac{Y_n}{2}\right) + \frac{h^2}{4}\left(1 + \frac{Y_n}{2}\right),$$

together with $Y_0 = 0$. Then, using $h_1 = 0.2$, we have $Y_1^{h_1} = 0.21$, $Y_2^{h_1} = 0.44205$. On the other hand, using $h_2 = 0.1$, we have $Y_1^{h_2} = 0.1025$, $Y_2^{h_2} = 0.21025$, $Y_3^{h_2} = 0.32353$, $Y_4^{h_2} = 0.44261$.

ii) We have

$$E^{h_1}_{t=0.4} = \left|Y^{h_1}_{t=0.4} - y(0.4)\right| = 7.55516 \times 10^{-4}$$

$$E^{h_2}_{t=0.4} = \left|Y^{h_2}_{t=0.4} - y(0.4)\right| = 1.95516 \times 10^{-4}.$$

In addition, $h_2 = h_1/2$. Thus

$$E^{h_2}_{t=0.4} \sim h_2^p = \left(\frac{h_1}{2}\right)^p \sim \frac{E^{h_1}_{t=0.4}}{2^p}$$

where p is the order of the method. Using the logarithm, we finally get $p \approx 1.95$, which suggests that the desired order is 2.
