Cálculo Diferencial Aplicado

FINAL EXAM 2

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Problem 1 (2.0 mark) Consider the differential equation $x^2y'' - 3xy' + 4y = \ln x\,, \quad x > 0\,.$

(i) Apply a change of variable to transform it into an equation with constant coefficients.

(ii) Solve the obtained equation together with the conditions $y(1) = 1/2\,, \quad y'(1) = 1\,.$

SOLUCIÓN:

(i) After applying the change of variable $x = e^{\,t}$, the given differential equation becomes

$$\frac{d^2y(t)}{dt^2} - 4\frac{dy(t)}{dt} + 4y(t) \;=\; t\,,$$

which has constant coefficients.

(ii) The characteristic equation of the corresponding homogeneous ODE has a unique root, namely $r = 2$. Hence, the general solution of this ODE is given by

$$y_h(t) = c_1e^{\,2t} + c_2te^{\,2t}\,.$$

A particular solution of the nonhomogeneous ODE can be found as $y_p(t) = At + B$, where $A = 1/4$ and $B = 1/4$. Then, the general solution of the equation at point (i) is

$$y(t) = c_1e^{\,2t} + c_2te^{\,2t} + \frac{1}{4}(t+1)\,.$$

Finally, going back to the original independent variable x, we get the general solution of the given differential equation as

$$y(x) = c_1x^2 + c_2x^2\ln x + \frac{1}{4}(\ln x + 1)\,.$$

Using the conditions $y(1) = 1/2$ and $y'(1) = 1$ yields $c_1 = 1/4$ and $c_2 = 1/4$. Hence, the final solution reads

$$y(x) = \frac{1}{4}\,(x^2 + x^2 \ln x + \ln x + 1) \quad \Longrightarrow \quad \boxed{y(x) = (x^2+1)(\ln x + 1)/4}\,.$$

Problem 2 (2.0 mark) Consider the following system of first-order differential equations

$$\begin{pmatrix} X_1'(t) \\ X_2'(t) \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$$

with $\alpha \in \mathbb{R}$ and $t > 0$.

(i) Find the value of α for which the qualitative behavior of the solutions of the system changes (*hint*: calculate the eigenvalues of the coefficients matrix in terms of α). Justify your answer.

(ii) Find the solution of the system for $\alpha = 1$ and $(X_1(0), X_2(0)) = (1, 0)$. In addition, calculate the distance $d(t)$ between the position $(0,0)$ and the position of a particle that moves according to the computed solution (*hint*: use formula $d(t) = \sqrt{X_1(t)^2 + X_2(t)^2}\,$).

SOLUCIÓN:

(i) The eigenvalues of the coefficients matrix are

$$r_1 = \sqrt{4 - 5\alpha}\,, \qquad r_2 = -\sqrt{4-5\alpha}\,.$$

If $\alpha < 4/5$ such eigenvalues are real with opposite signs, hence the solutions of the system are linear combinations of exponential functions.

On the other hand, if $\alpha > 4/5$ the eigenvalues are (purely imaginary) complex conjugates, hence the solutions of the system are periodic.

As a consequence, the qualitative behavior of the solutions changes for $\boxed{\alpha = 4/5}$.

(ii) For $\alpha = 1$ the eigenvalues and eigenvectors of the coefficients matrix are given by

$$r_1 = i \quad \Longrightarrow \quad \vec{u}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$r_2 = -i \quad \Longrightarrow \quad \vec{u}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}.$$

The general solution of the system is then

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants. If the mentioned particle starts from the position $(1,0)$ at $t = 0$, then c_1 and c_2 satisfy

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

namely $c_1 = 0$ and $c_2 = 1$. Finally, the desired distance is given by

$$\boxed{d(t) = \sqrt{X_1(t)^2 + X_2(t)^2} = \sqrt{4\sin^2 t + 4 \sin t \cos t + 1}}\,.$$

Problem 3 (2.0 mark) Solve the following initial value problem

$$\begin{cases} x^2 + e^{\,y} + (xe^{\,y} + \cos y)\, y' &= 0 \\ y(0) &= g(\pi/2)\,, \end{cases}$$

knowing that the function g verifies

$$g'(x) = \sin(x)\,, \quad g(0) = -1\,.$$

SOLUCIÓN:
We have that $g(x) = -\cos(x)$, so that $g(\pi/2) = 0 = y(0)$. On the other hand, the given differential equation is exact. Thus, there exists a function $F(x, y(x))$ such that $\partial F/\partial x = x^2 + e^{\,y}$, $\partial F/\partial y = xe^{\,y} + \cos y$, hence

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0\,.$$

The function F can be obtained by integrating $\partial F/\partial x$ as

$$F = \int \left(x^2 + e^{\,y}\right) dx = \frac{x^3}{3} + xe^{\,y} + \phi(y)\,.$$

In order to get the function $\phi(y)$, let us differentiate the previous expression with respect to y and equate the result to $\partial F/\partial y = xe^{\,y} + \cos y$, from what we get $\phi'(y) = \cos y$ and thus $\phi(y) = \sin y$ (taking the integration constant equal to zero). As a consequence

$$F = \frac{x^3}{3} + xe^{\,y} + \sin y\,.$$

From $dF/dx = 0$ we get $x^3/3 + xe^{\,y} + \sin y = C$. Furthermore, $y(0) = 0$ implies $C = 0$. Finally, the desired solution is

$$\boxed{\frac{x^3}{3} + xe^{\,y} + \sin y = 0}\,.$$

Problem 4 (2.0 mark) Consider the following model for the heat equation.

$$\begin{aligned} \text{Partial Differential Equation (PDE)} &: \quad \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t)\,, \ \ t > 0\,, \ \ x \in (0, \pi/3) \\ \text{Boundary Conditions (BC)} &: \quad \frac{\partial u}{\partial x}(0,t) = 0\,, \ \ \frac{\partial u}{\partial x}(\pi/3, t) = 0\,, \ \ t > 0 \\ \text{Initial Condition (IC)} &: \quad u(x,0) = f(x)\,, \ \ x \in [0, \pi/3]\,. \end{aligned}$$

Apply the separation of variables $u(x,t) = X(x)\,T(t) \not\equiv 0$.

(i) Prove that $X(x)$ satisfies the following boundary value problem

$$X'' + \lambda X = 0\,, \quad X'(0) = 0\,, \quad X'(\pi/3) = 0\,,$$

and find the values of the separation constant $\lambda \geq 0$ providing nonzero solutions.

(ii) Knowing that the solution $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-9n^2 t} \cos(3nx) \ , \quad \text{with } A_n \in \mathbb{R} \, ,$$

find the approximate value of $u(\pi/6, 1/9)$ by considering the first three terms only of the previous series and taking $f(x) = 2x + 1$.

Note. The following result may be useful.

Given $L > 0$ and $m \, , n \in \mathbb{N} \cup \{0\}$, we have $\displaystyle\int_0^L \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) dx = \begin{cases} 0\, ; \ m \neq n \\ L/2\, ; \ m = n \neq 0 \\ L\, ; \ m = n = 0\, . \end{cases}$

SOLUCIÓN:

(i) By applying the separation of variables, we obtain $\dfrac{T'}{T} = \dfrac{X''}{X} = -\lambda$, where λ is the separation constant. Hence $X'' + \lambda X = 0$. In addition, by using the BC we get

$$\frac{\partial u}{\partial x}(0,t) = X'(0)T(t) = 0 \implies X'(0) = 0 \, , \text{ holding } \forall t \text{ and } T(t) \not\equiv 0\, ;$$

$$\frac{\partial u}{\partial x}(\pi/3,t) = X'(\pi/3)T(t) = 0 \implies X'(\pi/3) = 0 \, , \text{ holding } \forall t \text{ and } T(t) \not\equiv 0\, .$$

Let us now distinguish two cases.

Case 1. $\lambda = 0$

$X'' = 0 \implies X(x) = c_1 x + c_2$ with $c_1 \, , c_2 \in \mathbb{R}$. As $X'(x) = c_1$, we have that $X'(0) = 0 = c_1 = X'(\pi/3)$. Hence, if $\boxed{\lambda = 0}$ then the function $X(x) = c_2 \neq 0$ is a nonzero solution of the problem.

Case 2. $\lambda > 0$

Let us take $\lambda = a^2$, with $a > 0$. The corresponding characteristic equation is then $r^2 + a^2 = 0 \implies r = \pm ia \, , \ i \in \mathbb{C}$. Hence

$$X(x) = c_1 \cos(ax) + c_2 \sin(ax)\, ; \quad X'(x) = -ac_1 \sin(ax) + ac_2 \cos(ax) \, , \text{ with } c_1 \, , c_2 \in \mathbb{R}\, .$$

Applying the BC we get $X'(0) = 0 \implies c_2 = 0$; $X'(\pi/3) = 0 \implies -ac_1 \sin(a\pi/3) = 0$. Then, imposing $c_1 \neq 0$ yields $\sin(a\pi/3) = 0 \implies a\pi/3 = n\pi \implies a = 3n \, , \ n = 1, 2, 3, \dots$. Finally, $\boxed{\lambda = (3n)^2 = 9n^2 \, , \ \ n = 1, 2, 3, \dots}$.

(ii) We need to calculate

$$u(\pi/6, 1/9) \approx A_0 + A_1 e^{-1} \cos(\pi/2) + A_2 e^{-4} \cos(\pi) = A_0 - \frac{A_2}{e^4}\, .$$

In order to compute the coefficients A_0 and A_2, let us apply the IC as to get

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos(3nx) = f(x) = 2x + 1\, .$$

In addition, using the orthogonality condition suggested in the note, we deduce that the coefficients A_n satisfy

$$A_0 = \frac{1}{L}\int_0^L f(x)dx = \frac{3}{\pi}\int_0^{\pi/3}(2x+1)dx = 1+\pi/3\,,$$

$$(n \geq 1) \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos(3nx)dx = \frac{6}{\pi}\int_0^{\pi/3}(2x+1)\cos(3nx)dx$$

$$\Longrightarrow A_2 = \frac{6}{\pi}\int_0^{\pi/3}(2x+1)\cos(6x)dx = \frac{1}{\pi}\left[(2x+1)\sin(6x)+\frac{1}{3}\cos(6x)\right]_0^{\pi/3} = 0\,.$$

Hence, the desired approximation is $\boxed{u(\pi/6, 1/9) \approx 1+\pi/3}$.

Problem 5 (2.0 mark) The following initial value problem

$$\begin{cases} y' &= t + \dfrac{y}{2} + 1 \\ y(0) &= 1 \end{cases}$$

must be numerically solved by using the Adams–Bashforth scheme

$$Y_{n+2} = Y_{n+1} + \frac{3}{2}\,h\,f(t_{n+1}, Y_{n+1}) - \frac{1}{2}\,h\,f(t_n, Y_n)\,.$$

(i) Calculate the approximated solution $Y^{h_1}_{t=0.3}$ of $y(0.3)$ with step $h_1 = 0.1$, knowing that Y_1 has to be computed by the explicit Euler method.

(ii) After noting that step $h_2 = 0.01$ yields the approximation $Y^{h_2}_{t=0.3} = 1.5327258$, estimate the order of the numerical scheme by means of $Y^{h_1}_{t=0.3}$, $Y^{h_2}_{t=0.3}$, and the exact solution given by $y(t) = 7e^{t/2} - 2(t+3)$.

SOLUCIÓN:

(i) From the initial condition we get $Y_0 = 1$, while the explicit Euler method provides $Y_1^{h_1} = 1.15$. Finally, the Adams–Bashforth scheme yields $Y_2^{h_1} = 1.32625$ and $\boxed{Y_3^{h_1} \equiv Y^{h_1}_{t=0.3} = 1.52197}$.

(ii) Let us calculate

$$E^{h_1}_{t=0.3} = \left|Y^{h_1}_{t=0.3} - y(0.3)\right| = 0.01087095 \quad \text{and} \quad E^{h_2}_{t=0.3} = \left|Y^{h_2}_{t=0.3} - y(0.3)\right| = 0.00011390\,.$$

As the reduction factor between steps h_1 and h_2 is $q = 10$, we have

$$E^{h_2}_{t=0.3} \approx h_2^p = \left(\frac{h_1}{10}\right)^p \approx \frac{E^{h_1}_{t=0.3}}{10^p}\,,$$

where p is the order of the method. After taking the logarithms, we get $p \approx 1.98$. Hence, we can estimate the order of the given numerical scheme as $\boxed{p = 2}$.
