# Applied Differential Calculus

**Self-Assessment: Test 1**

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**Problem 1** Consider the first-order differential equation

$$(3kx^2y + e^{\,y}) + (x^3 + kxe^{\,y} - 2yk^2)\,y' \,=\, 0\,,$$

where $k$ is a real parameter.

(a) Find the value of $k$ that makes the equation exact.

(b) Solve the equation for that value of $k$.

**SOLUTION:**
Let $M(x,y) = 3kx^2y + e^{\,y}$ and $N(x,y) = x^3 + kxe^{\,y} - 2yk^2$. Then, the equation is exact if

$$\frac{\partial M}{\partial y} \,=\, 3kx^2 + e^{\,y} \,=\, 3x^2 + ke^{\,y} \,=\, \frac{\partial N}{\partial x}\,,$$

which is satisfied if $k = 1$. As a consequence, for $k = 1$, there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} \,=\, M(x,y) \,=\, 3x^2y + e^{\,y}\,, \tag{1}$$

$$\frac{\partial \psi}{\partial y} \,=\, N(x,y) \,=\, x^3 + xe^{\,y} - 2y\,. \tag{2}$$

Now, integration of (3) with respect to $x$ yields $\psi(x,y) = yx^3 + xe^{\,y} + g(y)$, where $g(y)$ is a function to be determined. Then, equating the derivative with respect to $y$ of the previous expression with (4) provides

$$\frac{\partial \psi}{\partial y} \,=\, x^3 + xe^{\,y} + \frac{dg}{dy} \,=\, x^3 + xe^{\,y} - 2y\,,$$

hence

$$\frac{dg}{dy} \,=\, -2y \qquad \Longrightarrow \qquad g(y) \,=\, -y^2 + \alpha\,, \quad \alpha \in \mathbb{R}\,.$$

Using one of the possible expressions for $g(y)$ obtained for $\alpha = 0$, the general solution of the equation can be finally written as

$$\psi(x,y) = c \qquad \Longleftrightarrow \qquad \boxed{yx^3 + xe^{\,y} - y^2 = c}$$

where $c$ is an arbitrary constant.

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**Problem 2** Solve the following first-order differential equation

$$y = (x + \sqrt{xy})\, y'$$

for $x > 0$, together with the initial condition $y(1) = 1$.

**SOLUTION:**
This is a nonlinear, *homogeneous*, first-order differential equation. In order to solve it, we can use the change of variable $v = y/x$, which yields $xv' + v = y'$. Then

$$-vx + \left(x + \sqrt{x^2 v}\right)\left(v'x + v\right) = 0 \quad \Longrightarrow \quad v'\left(x^2 + x^2\sqrt{v}\right) + x\,v^{3/2} = 0$$

$$\Longrightarrow \quad v'\left(v^{-3/2} + \frac{1}{v}\right) + \frac{1}{x} = 0\,,$$

which is a separable equation in $v$. Thus, we get

$$-2\,v^{-1/2} + \ln|v| + \ln x = c\,,$$

where $c$ is an arbitrary constant. Finally, using $v = y/x$ and the initial condition $y(1) = 1$, an implicit expression for the desired solution is obtained as

$$\boxed{-2\sqrt{\frac{x}{y}} + \ln\frac{|y|}{x} + \ln x = -2}\,.$$

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**Problem 3** Consider the first-order differential equation

$$\big(\sin^2 x + 4xye^{xy^2} - x\big)\, y' + 2y\sin x\cos x + 2y^2 e^{xy^2} - y \;=\; 0\,.$$

(a) Classify the equation, justifying your answer.

(b) Find the general solution of the equation.

**SOLUTION:**

(a) Let $M(x,y) \;=\; 2y\sin x\cos x + 2y^2 e^{xy^2} - y$ and $N(x,y) \;=\; \sin^2 x + 4xye^{xy^2} - x\,.$ Then, the equation is exact as

$$\frac{\partial M}{\partial y} \;=\; 2\sin x\cos x - 1 + 4xy^3 e^{xy^2} + 4ye^{xy^2} \;=\; \frac{\partial N}{\partial x}\,.$$

(a) As a consequence, there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} \;=\; M(x,y) \;=\; 2y\sin x\cos x + 2y^2 e^{xy^2} - y\,, \tag{3}$$

$$\frac{\partial \psi}{\partial y} \;=\; N(x,y) \;=\; \sin^2 x + 4xye^{xy^2} - x\,. \tag{4}$$

Now, integration of (3) with respect to $x$ yields $\psi(x,y) = y\sin^2 x - xy + 2e^{xy^2} + g(y)$, where $g(y)$ is a function to be determined. Then, equating the derivative with respect to $y$ of the previous expression with (4) provides

$$\frac{\partial \psi}{\partial y} \;=\; \sin^2 x + 4xye^{xy^2} - x + \frac{dg}{dy} \;=\; \sin^2 x + 4xye^{xy^2} - x\,,$$

hence

$$\frac{dg}{dy} = 0 \qquad \Longrightarrow \qquad g(y) \;=\; \alpha\,, \quad \alpha \in \mathbb{R}\,.$$

Using one of the possible expressions for $g(y)$ obtained for $\alpha = 0$, the general solution of the equation can be finally written as

$$\psi(x,y) \;=\; c \qquad \Longleftrightarrow \qquad \boxed{y\sin^2 x - xy + 2e^{xy^2} \;=\; c}\,,$$

where $c$ is an arbitrary constant.

**Problem 4** Given the Ordinary Differential Equation (ODE):

$$-5x^4 + 2y + xy' = 0 \quad \text{with} \quad x > 0,$$

i) Classify this ODE.

ii) Solve the ODE with initial condition $y(1) = 2$.

**SOLUTION:**

i) First order linear non-homogeneous ODE.

ii) We first find an integrating factor, which is $x$: $x^2y' + 2xy = 5x^5$. (From $y' + 2\frac{y}{x} = 5x^3$, we find the integrating factor: $e^{2\int dx/x} = e^{2\ln x} = x^2$). Then $(x^2y)' = 5x^5$ and

$$x^2y = \frac{5}{6}x^6 + c \Longrightarrow 2 = \frac{5}{6} + c \Longrightarrow c = \frac{7}{6}.$$

We get

$$\boxed{y(x) = \frac{5}{6}x^4 + \frac{7}{6x^2}.}$$

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