Applied Differential Calculus

Self-assessment: Test 2

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Problem 1 A particular solution of $y'' + 2y' + y = e^{-t}\ln t$, for $t > 0$, is given by

$$y_p(t) = \frac{1}{2}\,t^2 e^{-t}\ln t - \frac{3}{4}\,t^2 e^{-t}.$$

Using this information, calculate the general solution of the following second-order differential equation

$$y'' + 2y' + y = e^{-t}\ln t + (1-t)\,e^{-t}.$$

SOLUTION:
The solution $y_H(t)$ of the homogeneous equation

$$y'' + 2y' + y = 0$$

associated with the given differential equation is obtained by finding the roots of the corresponding characteristic equation $r^2 + 2r + 1 = 0$, which are $r_1 = r_2 = -1$ (real and equal) and yield

$$y_H(t) = c_1 e^{-t} + c_2 t\,e^{-t},$$

where c_1 and c_2 are two arbitrary constants. Now, by the principle of superposition, a particular solution of the given equation can be calculated as $y_p(t) + y^*(t)$, where $y_p(t)$ is given above and $y^*(t)$ is a particular solution of

$$y'' + 2y' + y = (1-t)\,e^{-t}. \tag{1}$$

The latter can be found by the method of undetermined coefficients starting from the initial guess

$$y^*(t) = t^2(At + B)\,e^{-t}, \tag{2}$$

with A, $B \in \mathbb{R}$ to be determined. Thus, after substituting (4) into (3) and equating all coefficients, we get $A = -1/6$ and $B = 1/2$. Finally, the general solution of the given equation is given by

$$\boxed{y(t) = y_H(t) + y_p(t) + y^*(t) = c_1 e^{-t} + c_2 t\,e^{-t} + \tfrac{1}{2}\,t^2 e^{-t}\ln t - \tfrac{3}{4}\,t^2 e^{-t} + t^2\left(-\tfrac{1}{6}t + \tfrac{1}{2}\right)e^{-t}}\,.$$

Problem 2 Solve the following second-order differential equation

$$y'' + 3y' + 2y \;=\; \sin(e^{x})$$

and verify the obtained result.

SOLUTION:
The characteristic equation of the associated homogeneous equation has two roots, namely $r_1 = -1$ and $r_2 = -2$. Hence, its general solution is given by

$$y_h(x) = c_1 e^{-x} + c_2 e^{-2x}\,,$$

where c_1 and c_2 are two arbitrary constants. A particular solution of the nonhomogeneous equation can be found by the method of variation of parameters as

$$y_p(x) = u_1(x)\,e^{-x} + u_2(x)\,e^{-2x}\,,$$

where u_1 and u_2 are two functions to be found. Then, their derivatives must satisfy the system

$$\begin{cases} u_1'\,e^{-x} + u_2'\,e^{-2x} \;=\; 0 \\ -u_1'\,e^{-x} - 2\,u_2'\,e^{-2x} \;=\; \sin(e^{x})\,, \end{cases}$$

which yields

$$\begin{cases} u_1' \;=\; e^{x}\,\sin(e^{x}) \\ u_2' \;=\; -e^{2x}\,\sin(e^{x})\,. \end{cases}$$

Hence, upon integration we get

$$\begin{cases} u_1 \;=\; -\cos(e^{x}) \\ u_2 \;=\; e^{x}\,\cos(e^{x}) - \sin(e^{x})\,, \end{cases}$$

where u_2 is obtained integrating by parts. Thus, the general solution of the given differential equation reads

$$\boxed{y(x) \;=\; y_h(x) + y_p(x) \;=\; c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\,\sin(e^{x})}\;,$$

whose validity is easily verified by substituting its expression into the equation and checking that an identity is obtained.

Problem 3 Consider the following differential equation

$$(x-1)\,y'' + y' \;=\; 0$$

and assume that the solution is given by the power series $y(x) = \sum_{n=0}^{\infty} a_n\,x^n$.

(a) Find the recurrence relation satisfied by the coefficients a_n .

(b) Apply the initial conditions $y(0) = 0$ and $y'(0) = 1$, then write the first three non-zero terms of the used power series.

SOLUTION:

(a) Let $y(x) = \sum_{n=0}^{\infty} a_n\, x^n$, hence

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Substituting these series into the equation yields

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-1} - \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \;=\; 0\,.$$

Now, in order to have the power x^n in all series, let us make a change of index in each one of them, which gives

$$\sum_{n=1}^{\infty} (n+1)n\, a_{n+1}\, x^n - \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} (n+1)\, a_{n+1}\, x^n \;=\; 0\,,$$

namely

$$-2a_2 + a_1 + \sum_{n=1}^{\infty} \Big[(n+1)n\, a_{n+1} - (n+2)(n+1)\, a_{n+2} + (n+1)\, a_{n+1} \Big]\, x^n \;=\; 0\,.$$

Finally, equating to zero the coefficients of each power of x, we get

$$-2a_2 + a_1 \;=\; 0\,, \quad (n+1)^2\, a_{n+1} - (n+2)(n+1)\, a_{n+2} \;=\; 0\,,$$

which can be expressed as

$$\boxed{a_2 = \frac{a_1}{2}\,, \quad a_{n+2} = \frac{n+1}{n+2}\, a_{n+1}}$$

with $n = 1, 2, 3, \ldots$.

(b) Using the recurrence relation obtained in (a), we get

$$a_2 = \frac{a_1}{2}\,, \quad a_3 = \frac{a_1}{3}\,, \quad a_4 = \frac{a_1}{4}\,, \quad \ldots\,.$$

Since the initial conditions provide $a_0 = 0$ and $a_1 = 1$, we can finally write the first three non-zero terms of the used power series as

$$\boxed{x\,, \; \frac{x^2}{2}\,, \; \frac{x^3}{3}}\,.$$

Problem 4 Solve the initial value problem : $xy'' - (2x+1)y' + (x+1)y = 0$, with $y(1) = 0$, $y'(1) = e$ knowing that, obviously, $y = e^x$ is a solution of the homogeneous ODE.

SOLUTION:
Substituting $y = e^x u(x)$ in the ODE, we get

$$xe^x(u'' + 2u' + u) - (2x+1)e^x(u' + u) + (x+1)e^x u = 0,$$

i.e., $e^x(xu'' - u') = 0$. Then $xu'' - u' = 0$. Let $v = u'$, we get $xv' = v \Longrightarrow v'/v = 1/x$. Integrating once, we find $\ln v = \ln x + c$ or $v = e^c x$. This is equivalent to $u' = 2c_1 x$ with $2c_1 = e^c$ an arbitrary constant. Integrating once again, we obtain $u = c_1 x^2 + c_2$, in which c_1 and c_2 are constant. Then $y = (c_1 x^2 + c_2)e^x$. The initial conditions yield $y(1) = (c_1 + c_2)e = 0$ or $c_1 = -c_2$ and $y'(1) = (2c_1 + c_1 + c_2)e = e$. We find $2c_1 = 1$ of $c_1 = -c_2 = 1/2$. The solution of the IVP is $y(x) = \frac{1}{2}(x^2 - 1)e^x$.
A different way to find the solution is to use Abel's formula after rewriting the equation as $y'' - (2x+1)y'/x + (x+1)y/x = 0$. Notice that minus the coefficient of y' is $2 + 1/x$. Then the Wronskian is

$$W = e^{\int (2+1/x)dx} = e^{2x} e^{\ln x} = x\, e^{2x}.$$

Now we obtain the other independent solution from the formula

$$W = \begin{vmatrix} e^x & y \\ e^x & y' \end{vmatrix} = e^x(y' - y) = x\, e^{2x} \Longrightarrow e^{-x}(y' - y) = x \Longrightarrow (e^{-x}y)' = x \Longrightarrow y = \frac{x^2}{2}e^x.$$

Thus the general solution is $y = (c_1 x^2 + c_2)e^x$, as before.