Applied Differential Calculus

Self-assessment: Test 3

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Problem 1 Solve the following system of first-order differential equations

$$\begin{cases} x_1' &= -4x_1 + 2x_2\,, \quad x_1(0) = 2 \\ x_2' &= -\dfrac{5}{2}x_1 + 2x_2\,, \quad x_2(0) = -3\,. \end{cases}$$

SOLUTION:
The solution of the given system is found by first calculating the eigenvalues and eigenvectors of the coefficients matrix

$$A = \begin{bmatrix} -4 & 2 \\ -5/2 & 2 \end{bmatrix},$$

which are given by

$$r_1 = 1 \quad \Longrightarrow \quad \boldsymbol{u}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$r_2 = -3 \quad \Longrightarrow \quad \boldsymbol{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since the eigenvalues are real and different, the corresponding eigenvectors are linearly independent, hence the general solution of the system can be written as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{\,t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t},$$

where c_1 and c_2 are two arbitrary constants. Then, initial conditions yield $c_1 = -1$ and $c_2 = 2$, hence

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -\begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{\,t} + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}}\,.$$

Problem 2 (i) Prove that the differential equation $y'' - 6y' + 13y = 0$ is equivalent to the following system of first-order differential equations

$$X' = \begin{bmatrix} 0 & 1 \\ -13 & 6 \end{bmatrix} X, \quad \text{where} \quad X = X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

(ii) Solve the system in (i) knowing that

$$X(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

SOLUTION:

(i) Let us apply the following change of variables

$$y(t) = x_1(t), \qquad y'(t) = x_2(t).$$

Hence, $x_1' = y' = x_2$ and $x_2' = y'' = -13y + 6y' = -13x_1 + 6x_2$. Writing this result in matrix-vector form yields the given system of first-order differential equations.

(ii) In order to solve the system in (i), let us calculate the eigenvalues of the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 \\ -13 & 6 \end{bmatrix},$$

which are $r_1 = 3 + 2i$ and $r_2 = 3 - 2i$. The associated eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 3 - 2i \\ 13 \end{pmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{pmatrix} 3 + 2i \\ 13 \end{pmatrix},$$

respectively. Using r_1 and $\vec{v}_1$, the general solution of the system can be expressed as

$$X(t) = c_1 \operatorname{Re}\left\{\begin{pmatrix} 3 - 2i \\ 13 \end{pmatrix} e^{(3+2i)\,t}\right\} + c_2 \operatorname{Im}\left\{\begin{pmatrix} 3 - 2i \\ 13 \end{pmatrix} e^{(3+2i)\,t}\right\},$$

where c_1 and c_2 are two arbitrary constants. Hence

$$X(t) = e^{3t} \begin{pmatrix} (3c_1 - 2c_2)\cos(2t) + (2c_1 + 3c_2)\sin(2t) \\ 13c_1 \cos(2t) + 13c_2 \sin(2t) \end{pmatrix}.$$

Finally, the initial condition gives $X(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3c_1 - 2c_2 \\ 13c_1 \end{pmatrix}$, namely $c_1 = 2/13$ and $c_2 = -10/13$. Thus, the desired solution is $\boxed{X(t) = e^{3t} \begin{pmatrix} -2\sin(2t) + 2\cos(2t) \\ -10\sin(2t) + 2\cos(2t) \end{pmatrix}}$.

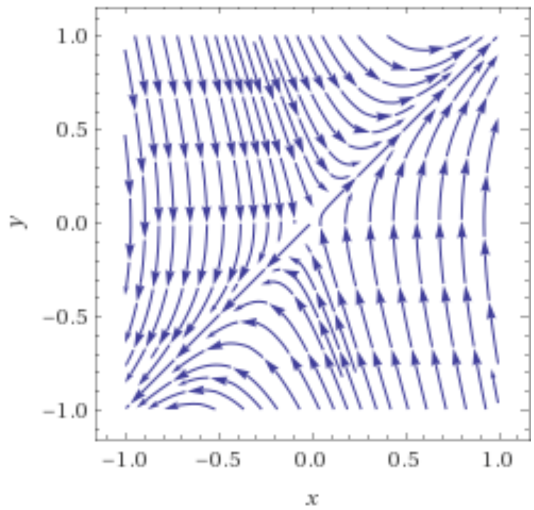
Problem 3 Given the ODE: $y''+2y'-3y=0$, write it as a system of two first order ODEs, classify the equilibrium point $(0,0)$ and draw its phase portrait, indicating explicitly any real eigendirections that may be relevant.

SOLUTION:
Letting $y=x_1$, $y'=x_2$, we find $x_1'=x_2$ and $x_2'=3x_1-2x_2$, i.e.,

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{\underline{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

As $\mathrm{Tr}\underline{\underline{A}}=-2$ and $\det\underline{\underline{A}}=-3$, we find the characteristic polynomial $\lambda^2+2\lambda-3=0$ that produces the eigenvalues $\lambda_1=1$ and $\lambda_2=-3$, one positive and one negative. Thus $(0,0)$ is a saddle point.

The eigenvectors have components satisfying $-v_1+v_2=0$ for $\lambda_1=1$ and $3v_1+v_2=0$ for $\lambda_2=-3$. These expressions yield $v_1=v_2=1$ and $v_1=1$, $v_2=-3$, respectively. Thus the corresponding $\boxed{\text{eigendirections are } (1,1) \text{ for } \lambda_1=1 \text{ and } (1,-3) \text{ for } \lambda_1=-3}$. The phase portrait for the origin is depicted in the figure.



Problem 4 Solve the following system of first order linear ODEs with initial condition $x_1(0) = 1$, $x_2(0) = 1$:

$$\begin{cases} x_1' = 2x_1 - 3x_2 \\ x_2' = 6x_1 - 4x_2 \end{cases}$$

SOLUTION:

The coefficient matrix has trace -2 and determinant 10. Therefore the characteristic polynomial is $\lambda^2+2\lambda+10 = (\lambda+1)^2+9$ and thus the eigenvalues are $-1\pm 3i$. The components of the eigenvector corresponding to $-1+3i$ satisfy: $(3-3i)v_1 - 3v_2 = 0$, so that $v_1 = 1$ and $v_2 = 1-i$. Then the general solution of the system of ODEs is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Re\left\{ e^{-t+i3t}(a+ib)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}\right\} = e^{-t} Re\left\{\begin{pmatrix} e^{i3t}(a+ib) \\ e^{i3t}(a+ib)(1-i) \end{pmatrix}\right\}.$$

Since $(a+ib)(1-i) = a+b+i(b-a)$, we find

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} a\cos(3t) - b\sin(3t) \\ (a+b)\cos(3t) + (a-b)\sin(3t) \end{pmatrix}.$$

The initial condition produces $1 = a$ and $1 = a - b$, so $b = 0$, and the solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{-t}\begin{pmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{pmatrix}.}$$
