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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS

Degree in Biomedical Engineering

Chapter 1

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General index

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FIRST ORDER DIFFERENTIAL EQUATIONS

We start with a chapter dedicated to the first order differential equations, the most important resolution methods and their application to the study of geometrical, physical or social sciences problems.

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1.1 Introduction

An **ordinary differential equation** (in short **ODE**) is a relation of some derivatives of a function. They appear in many contexts, such as physics, chemistry, engineering, sociology, and so on.

***Example* 1.** The free fall of an object is described with the equation:

$$my'' = mg \quad \Longrightarrow \quad y = \frac{g}{2}t^2 + v_0 t + y_0,$$

where $y = y(t)$ is the height of the object at time t, m is the mass and g is the acceleration of gravity. We obtain the height in terms of the initial position y_0 and the initial velocity v_0.

In this first example it was enough to integrate twice the ODE we are given, $y'' = g$.

***Example* 2.** Following **Newton's law of cooling**, when the difference of temperatures between a body and the environment is not too big, the heat that is transferred is proportional to the difference of temperatures. We have the equation:

$$T'(t) = -k(T - T_m),$$

where $T = T(t)$ is the temperature at time t, k is a positive constant that describes the heat diffusivity and T_m is the temperature of the environment. This is a second example of ODE that we will learn how to solve.

***Example* 3.** **Malthus' law of demography** establishes that the growth rate is proportional to the population that exists, $P = P(t)$. Taking into account the negative effects derived from overcrowding we have to add a decaying term. We obtain the ODE

$$P'(t) = aP - bP^2.$$

that is called **logistic equation**, where a and b are positive constants.

In general, an ordinary differential equation of order n is an expression of the form:

$$F(x, y(x), y'(x), \cdots, y^{n)}(x)) = 0,$$

for some function F. The **order** of the equation is that of the highest order derivative that appears in the expression. Its solution, $y = y(x)$, contains n arbitrary constants, as many as the order. So, the solution is a **family of**

curves, that we call **integral curves** or simply **solutions**. The solutions can be in an implicity form, and in this case we do not obtain an expression for $y(x)$. To obtain the value of the constants we need data, one for each constant. Using the data we identify in the family of integral curves the one that goes through a determined point. The set of the equation together with the data is called **problem**, for example:

$$\begin{cases} y'' + 4y = 0, \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

When we have a family of curves, we can obtain the differential equation of which it is the solution by deriving as many times as constants in the family and removing the constants.

***Example* 4.** Starting with the family of circumferences with center at the origin we obtain the ODE

$$x^2 + y^2 = c^2 \quad \Longrightarrow \quad 2x + 2yy' = 0 \quad \Longrightarrow \quad y' = -\frac{x}{y}.$$

In this chapter we concentrate in the study of first order equations, that is, expressions of the form

$$F(x, y, y') = 0,$$

and suppose always that we can clear y' in the previous identity. So the equations will have the form

$$y' = f(x, y). \tag{1.1}$$

Also, it is useful to write this equation in its differential form with the notation

$$M(x, y)dx + N(x, y)dy = 0. \tag{1.2}$$

It is straightforward to change from one notation to the other and we will use the most appropriate to the case.

***Example* 5.** The equation of the previous example can be written as

$$xdx + ydy = 0.$$

Once we solve the equation, the solution of the problem with the data may not be unique, but we are sure that in some cases it is:

Theorem 1.1. ***(Picard)***

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on a closed rectangle $R \subset \mathbb{R}^2$, then through every point of the interior of R there is one and only one integral curve of the equation

$$y' = f(x, y).$$

1.2 Elementary methods

1.2.1 Separate variables equations

These can be written in the form $y' = g(x)h(y)$. They are the simplest and are solved by separating variables:

$$\frac{dy}{h(y)} = g(x)dx$$

and integrating both sides of the equality, each one with respect to the variable that appears in it. We join the integrating constants in only one and finally we simplify. If we have the differential form (1.2) we must have $M(x,y) = M(x)$, $N(x,y) = N(y)$.

***Example* 6.**

$$y' = x^2 y \implies \frac{dy}{y} = x^2 dx \implies \log|y| = \frac{x^3}{3} + C \implies y = K\mathrm{e}^{x^3/3}.$$

1.2.2 Changes of variables

With the new variable $z = A(x,y)$, the ODE $y' = f(x,y)$ in the variables (x,y) becomes an ODE in the variables (z,x),

$$z' = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} y'.$$

The idea is to obtain a new equation that is easier to solve.

***Example* 7.** The equations of the form $y' = f(ax+by+c)$ are solved with the change of variable $z = ax + by + c$, that reduces them to separate variables equations, $z' = a + bf(z)$.

1.2.3 Homogeneous equations

We say that a function of two variables f is **homogeneous of degree** α if

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x,y), \qquad \forall x,y \in \mathbb{R}, \quad \lambda > 0.$$

An equation $y' = f(x,y)$ is called **homogeneous** if f is homogeneous of degree 0. It can be solved with the change of variable $z = \dfrac{y}{x}$, that reduces it to an equation with separate variables in the variables (x,z).

***Example* 8.**

$$y' = \frac{x+y}{x-y}, \qquad z = \frac{y}{x} \quad \Longrightarrow \quad y' = xz' + z \quad \Longrightarrow \quad \frac{1-z}{1+z^2}\,dz = \frac{dx}{x}$$

$$\operatorname{arctg} z - \frac{1}{2}\log(1+z^2) = \log|x| + K \quad \Longrightarrow \quad \operatorname{arctg}\frac{y}{x} = \log\sqrt{x^2+y^2} + K.$$

1.2.4 Exact equations

An equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is **exact** if there exists a function f such that

$$M(x,y) = \frac{\partial f}{\partial x}, \qquad N(x,y) = \frac{\partial f}{\partial y},$$

and the expression $Mdx+Ndy$ is called an **exact differential**. The function f is the potential of the vector field $\vec{F} = (M,N)$, since it satisfies $\nabla f = \vec{F}$.

Proposition 1.2.

A differential equation $Mdx + Ndy = 0$ *is exact if and only if* $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. *In this case the general solution is* $f(x,y) = C$, *where* f *is the potential of the vector field* (M,N).

This result can be read as: the potential exists if and only if the vector field is conservative.

***Example* 9.** $e^y dx + (xe^y + 2y)dy = 0$ is exact because $\dfrac{\partial(e^y)}{\partial y} = \dfrac{\partial(xe^y+2y)}{\partial x} = e^y$. We look for the solution $f(x,y) = C$, where:

$$\begin{cases} \dfrac{\partial f}{\partial x} = e^y \implies f(x,y) = xe^y + K(y) \\ \dfrac{\partial f}{\partial y} = xe^y + 2y \end{cases} \implies xe^y + 2y = xe^y + K'(y),$$

so, $K(y) = y^2 + c$. Finally $f(x,y) = xe^y + y^2 + c$, and the solution is $xe^y + y^2 = C$.

Some exact easy-to-recognize exact differentials are:

$$d(xy) = x\,dy + y\,dx\,, \qquad d(x^2+y^2) = 2(x\,dx + y\,dy)\,,$$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}\,, \qquad d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}\,.$$

1.2.5 Integrating factors

A non exact equation of the form $Mdx + Ndy = 0$ can be multiplied by an arbitrary non-zero function, $\mu(x, y)$, leading to the equivalent equation

$$\mu M dx + \mu N dy = 0.$$

The function μ is called **integrating factor** if this last equation is exact, that is, if

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M).$$

There is no general formula to find an integrating factor, but usually we seek for the simplest forms (for example, a function of only one variable) and then we solve the equation. Only **one factor** is necessary, so when we find μ no integrating constants are needed.

***Example* 10.** We check that the following equation is not exact and seek for an integrating factor $\mu = \mu(x)$:

$$ydx + (x^2y - x)dy = 0 \quad\Longrightarrow\quad \mu(x)ydx + \mu(x)(x^2y - x)dy = 0$$

that must be exact, then:

$$\frac{\partial}{\partial y}(\mu(x)y) = \frac{\partial}{\partial x}(\mu(x)(x^2y - x)) \quad\Longrightarrow\quad \frac{\mu'(x)}{\mu(x)} = \frac{-2}{x} \quad\Longrightarrow\quad \mu(x) = \frac{1}{x^2}.$$

The new equation: $\frac{y}{x^2}dx + \left(y - \frac{1}{x}\right)dy = 0$, is exact, so the solution is $f(x, y) = C$, where:

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{y}{x^2} \implies f(x,y) = -\dfrac{y}{x} + C(y) \\ \dfrac{\partial f}{\partial y} = y - \dfrac{1}{x} \end{cases} \quad\Longrightarrow\quad y - \frac{1}{x} = -\frac{1}{x} + C'(y);$$

Therefore, $C(y) = \frac{y^2}{2} + c$ and the solution is $\frac{-y}{x} + \frac{y^2}{2} = K$.

1.2.6 Linear equations

An equation is **linear** if the highest order derivative is a linear function of the lower order derivatives. The **canonical form** of a linear equation of first order is:

$$y' + P(x)y = Q(x).$$

This can be solved using an integrating factor that depends only on the variable x, and that turns out to be $\mu(x) = \mathrm{e}^{\int P(x)\,dx}$.

The general solution has the form:

$$y = \mathrm{e}^{-\int P(x)\,dx}\left[\int Q(x)\mathrm{e}^{\int P(x)\,dx}\,dx + C\right].$$

***Example* 11.** $xy' - 4y = x^3$ is a linear equation of canonical form $y' - \frac{4}{x}\,y = x^2$, the solution is (for $C \in \mathbb{R}$):

$$y = \mathrm{e}^{\int \frac{4}{x}\,dx}\left[\int x^2\mathrm{e}^{-\int \frac{4}{x}\,dx}dx + C\right] = x^4\left[\int x^{-2}dx + C\right] = -x^3 + Cx^4.$$

1.3 Other kinds of equations

1.3.1 Bernoulli equations

These are the equations that can be written in the form:

$$y' + P(x)y = Q(x)y^n, \qquad n \neq 0, 1.$$

This can be solved with the **change of variable** $z = y^{1-n}$, that transforms it into a linear equation for $z(x)$.

***Example* 12.** $x^2y' + xy = x^3y^3$ is of Bernoulli type with $n = 3$:

$$z = y^{-2} \Longrightarrow z' = -2y^{-3}y' \Longrightarrow y' = -\frac{y^3z'}{2};$$

we make the substitution into the equation and simplify:

$$-\frac{y^3z'}{2}x^2 + xy = x^3y^3 \Longrightarrow z' - \frac{2}{x}z = 2x;$$

now we solve as a linear equation and undo the change:

$$z = \mathrm{e}^{\int \frac{2}{x}\,dx}\left[\int 2x\mathrm{e}^{-\int \frac{2}{x}\,dx}dx + C\right] = x^2\left[\int \frac{2}{x}\,dx + C\right] = x^2(\log(x^2) + C)$$

$$\Longrightarrow y = \frac{\pm 1}{x\sqrt{\log(x^2) + C}}.$$

1.3.2 Riccati equations

They have the form

$$y' = A(x) + B(x)\,y + C(x)\,y^2.$$

If a particular solution for this equation is known, $y_1(x)$, (given by the problem or obtained by some other method) the general solution can be written as $y = y_1 + z$, where z is the solution of a Bernoulli equation:

$$y_1' + z' = A + B(y_1 + z) + C(y_1 + z)^2 \implies$$

$$y_1' + z' = A + By_1 + Bz + Cy_1^2 + 2Cy_1 z + Cz^2 \implies$$

$$z' = (B + 2Cy_1)z + Cz^2.$$

1.3.3 Order reduction

There are special second order equations, $F(x, y, y', y'') = 0$, that can be solved with first order techniques:

Case 1. If the variable y does not appear: The second order equation $F(x, y', y'') = 0$ becomes a first order equation with the change of variable $y'(x) = p(x)$, which implies $y''(x) = p'(x)$, leading to an equation in (x, p),

$$F(x, p, p') = 0.$$

Case 2. If the variable x does not appear: The second order equation $F(y, y', y'') = 0$ becomes a first order equation with the change of variable:

$$y'(x) = p(y) \quad \rightsquigarrow \quad y''(x) = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\,\frac{dp}{dy}\,,$$

leading to an equation in (y, p),

$$F(y, p, pp') = 0.$$

These methods can be used also to reduce the order in higher order equations.

***Example* 13.** In the problem

$$\begin{cases} y'' = y'\mathrm{e}^y, \\ y(0) = 0,\ y'(0) = 1, \end{cases}$$

there is no x; with the change $y' = p(y)$ we obtain:

$$\begin{cases} pp' = p\mathrm{e}^y, \\ p(0) = 1. \end{cases}$$

This equation implies $p = 0$ or $p' = e^y$. The first option does not verify the data; from the second we have $p = e^y + C$, where the datum implies $C = 0$. We integrate again, since $p = y'(x)$,

$$e^{-y} dy = dx \implies -e^{-y} = x + D.$$

With the datum $y(0) = 0$ we obtain that $y = -\log(1 - x)$.

1.4 Applications

1.4.1 Orthogonal trajectories

Two families of curves are **orthogonal** if every intersection of two curves, one of each family, is orthogonal. The question is, given a family of curves, how to find the set of the **orthogonal trajectories**. If the ODE of two families of curves are

$$y' = f(x, y), \qquad y' = g(x, y),$$

the families are orthogonal if $fg = -1$. So, the family of orthogonal trajectories to the family with ODE $y' = f(x, y)$ satisfies the equation

$$y' = \frac{-1}{f(x, y)}.$$

***Example* 14.** The differential equation of the family $x^2 + y^2 = 2cx$ is

$$y' = \frac{y^2 - x^2}{2xy},$$

then the orthogonal trajectories satisfy the equation

$$y' = \frac{-2xy}{y^2 - x^2},$$

that is homogeneous, and has the soluion: $x^2 + y^2 = Ky$.

1.4.2 Newton's law of cooling

We have seen that the ODE that describes the change of temperature of a body in a medium is

$$T'(t) = -k(T - T_m),$$

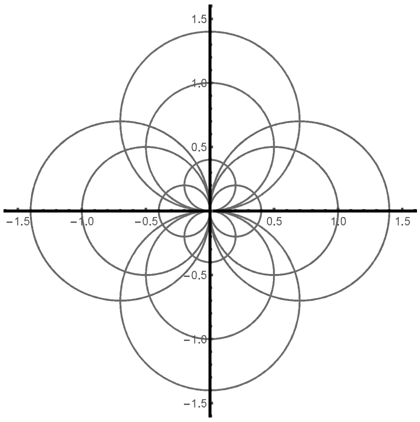

Figure 1.1: The orthogonal families of curves of example 14.

By separating variables and using the initial condition $T(0) = T_0$ we obtain:

$$T(t) = T_m + (T_0 - T_m)\mathrm{e}^{-Kt}.$$

If the inital temperature is less than (respectively greater than) the temperature of the environment the temperature will increase (respectively decrease) approaching the temperature of the environment in an asymptotic form.

1.4.3 Radioactive decay

It consists in the transformation of instable atoms (radium, uranium, cesium,...) in other kind of atoms. Approximately, the decay speed of a substance is proportional to the amount of the existing substance. The ODE for the rate of **disintegrations per minute** is

$$d'(t) = -kd(t).$$

The method of dating ancient objects is based in determining the rate of disintegration of certain substances comparing it with the initial disintegration rate, using some radioactive isotopes of slow disintegration. If we know what is called the **half-life** σ, that is, the necessary time to reduce the disintegration rate to the half, we obtain:

$$d(t) = d(0)2^{-t/\sigma},$$

from this we can compute $T = \sigma \log_2(d(0)/d(T))$ using the initial datum $d(T)$.

1.4.4 Population dynamics

Malthus' law of demography establishes the ODE for the amount of population at each time:

$$P'(t) = aP - bP^2,$$

starting with an initial population of $P(0) = P_0$. This equation can be solved separating variables or as a Bernoulli equation. The solution is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

We find that the population grows if $a - bP_0 > 0$, and decreases otherwise, and that the long-term equilibrium population is

$$\lim_{t\to\infty} P(t) = \frac{a}{b}.$$

These properties can also be obtained directly from the equation, since $P'(t) = 0$ implies $P(t) = 0$ or $P(t) = a/b$, and that $P'(t) > 0$ if and only if $a - bP(t) > 0$, that is, when $P(t) < a/b$.

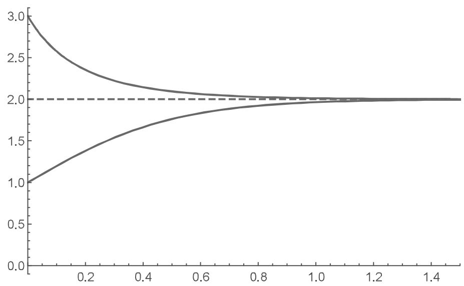

Figure 1.2: Population dynamics depending on the initial population.

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