uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS

Degree in Biomedical Engineering

Chapter 2

Elena Romera

Open Course Ware, UC3M
http://ocw.uc3m.es/matematicas

2

LINEAR EQUATIONS OF HIGHER ORDER

We present here resolution methods for linear differential equations of order higher than one, including the cases of constant and variable coefficients, together with some applications.

Contents

2.1 Second order linear equations

A linear **full equation** of second order has the general form

$$y'' + P(x)\,y' + Q(x)\,y = R(x).$$

P, Q and R are functions of x. We only know how to solve these equations in some specific cases, but not in general. Here we study the case in which P and Q are constants and some examples with non constant functions, but there is no general form to solve these last cases. The equation has two derivatives, so the general solution contains two constants. We say that the equation is **homogeneous** if $R(x) = 0$.

Theorem 2.1.

If $P(x)$, $Q(x)$ and $R(x)$ are continuous functions on a closed interval $[a, b]$, $x_0 \in [a, b]$ and y_0, y_0' are numbers, the equation

$$y'' + P(x)\,y' + Q(x)\,y = R(x)$$

has one and only one solution $y(x)$ on $[a, b]$ such that

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'\,.$$

If the data are given at different points the theorem does not hold anymore.

***Example* 15.** The following problem

$$\begin{cases} y'' + y = 0, \\ y(0) = 0, \quad y(1) = 0, \end{cases}$$

has no solution, while the problem

$$\begin{cases} y'' + y = 0, \\ y(0) = 0, \quad y(\pi) = 0, \end{cases}$$

has infinite solutions, $y(x) = k \sin x$, $k \in \mathbb{R}$.

Any homogeneous equation always has the trivial solution, $y = 0$. Besides:

Theorem 2.2.

If y_1 and y_2 are solutions of the homogeneous equation, then also is a solution:

$$y = c_1 y_1 + c_2 y_2, \qquad c_1, c_2 \in \mathbb{R}.$$

The set of solutions of a linear second order ODE is a vectorial space of dimension 2. Thus, two linearly independent solutions, that is, one is not a constant times the other, form a basis of this space.

Theorem 2.3.

If y_1 and y_2 are two linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0,$$

on the interval $[a, b]$, then the general solution on that interval is

$$y_h = c_1 y_1 + c_2 y_2, \qquad c_1, c_2 \in \mathbb{R}.$$

If y_p is a particular solution of the non-homogenous equation

$$y'' + P(x)\, y' + Q(x)\, y = R(x)\,,$$

the general solution of this last equation is:

$$y = y_p + y_h = y_p + c_1 y_1 + c_2 y_2, \qquad c_1, c_2 \in \mathbb{R}.$$

To determine the linear independence of two functions y_1, y_2, we use the **wronskian**, that is the following determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Lema 2.4.

Two functions are linearly dependent if and only if their wronskian is zero. Besides, if they are solutions of the homogeneous equation $y'' + Py' + Qy = 0$, then $W = c\mathrm{e}^{-\int P}$ for some constant $c \in \mathbb{R}$.

If we only know one solution of the homogeneous equation, we can find another linearly independent by:

Lema 2.5. ***(Abel's formula)***

If y_1 is a solution of $y'' + Py' + Qy = 0$, a solution that is linearly independent of y_1 is $y_2 = vy_1$, where

$$v(x) = \int \frac{1}{y_1^2(x)}\, \mathrm{e}^{-\int P(x)dx} dx.$$

2.1.1 Homogeneous equations with constant coefficients

The equations we consider now are of the kind

$$y'' + py' + qy = 0, \qquad p, q \in \mathbb{R}.$$

We seek for solutions of the form $y = \mathrm{e}^{rx}$, we substitute and find that r must be a root of the so named **characteristic equation**:

$$r^2 + pr + q = 0 \quad \Longrightarrow \quad r = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

There are three cases:

1. **Two different real roots.** If $p^2 - 4q > 0$, and r_1, r_2 are the roots, the solutions are $y_1 = \mathrm{e}^{r_1 x}$, $y_2 = \mathrm{e}^{r_2 x}$, that are independent. The general solution is:
$$y(x) = c_1 \mathrm{e}^{r_1 x} + c_2 \mathrm{e}^{r_2 x}.$$

2. **Two complex conjugated roots.** If $p^2 - 4q < 0$, The roots are $r_1 = a + ib$, $r_2 = a - ib$ and the associated solutions are
$$\mathrm{e}^{(a+ib)x} = \mathrm{e}^{ax}(\cos bx + i \sin bx), \qquad \mathrm{e}^{(a-ib)x} = \mathrm{e}^{ax}(\cos bx - i \sin bx).$$
In order to obtain real functions we consider instead linear combinations of these solutions that yield the real and imaginary parts:
$$\mathrm{e}^{ax} \cos bx = \frac{\mathrm{e}^{(a+ib)x} + \mathrm{e}^{(a-ib)x}}{2}, \qquad \mathrm{e}^{ax} \sin bx = \frac{\mathrm{e}^{(a+ib)x} - \mathrm{e}^{(a-ib)x}}{2i}.$$
These functions are linearly independent solutions. The general solution is then:
$$y(x) = \mathrm{e}^{ax}(c_1 \cos bx + c_2 \sin bx).$$

3. **One double root.** When $p^2 - 4q = 0$ we only obtain one solution, $y_1(x) = \mathrm{e}^{-px/2}$, but a second linearly independent solution is $y_2(x) = x\mathrm{e}^{-px/2}$. The general solution has the form:
$$y(x) = (c_1 + c_2 x)\mathrm{e}^{-px/2}.$$

***Example* 16.** In $y'' + 4y' + 4y = 0$, the characteristic equation is $r^2 + 4r + 4 = 0$, with double root $r = -2$, the solution is:

$$y = (c_1 + c_2 x)\mathrm{e}^{-2x}.$$

2.1.2 Method of undetermined coefficients

This method is useful to look for particular solutions of non-homogeneous equations when the independent term $R(x)$ has a special form: an exponential, sine or cosine or any of these multiplied by a polynomial.

1. **If** $R(x) = \mathrm{e}^{ax}$, we take $y_p(x) = A\mathrm{e}^{ax}$ if a is not a root of the characteristic polynomial (otherwise it would be a solution of the homogeneous equation). If it is a root, we take $y_p(x) = Ax\mathrm{e}^{ax}$. If this is also a solution of the homogeneous equation we take then $y_p(x) = Ax^2\mathrm{e}^{ax}$.

2. **If** $R(x) = K_1 \sin bx + K_2 \cos bx$, with K_1, K_2 constants, we consider
$$y_p(x) = A \sin bx + B \cos bx$$
if ib is not a root of the characteristic polynomial, in that case we take
$$y_p(x) = x(A \sin bx + B \cos bx).$$

3. **If** $R(x)$ **is a polynomial of degree** n, we take a polynomial of the same degree n:
$$y_p(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n.$$

4. **If** $R(x)$ is an exponential, a sine or a cosine multiplied by a polynomial, we look for a particular solution of the same form, using the previous ideas.

5. **If** $R(x)$ is a sum of functions of the previous types, by linearity we seek for a particular solution as a sum of functions, following the previous cases.

***Example* 17.** The equation $y'' - y' - 6y = 12x + 20\mathrm{e}^{-2x}$ has the following roots of the characteristic equation: $r_1 = 3$, $r_2 = -2$; so, we seek for a particular solution os the form $y_p(x) = Ax + B + Cx\mathrm{e}^{-2x}$, and obtain $A = -2$, $B = 1/3$, $C = -4$. The general solution is then
$$y(x) = c_1\mathrm{e}^{3x} + \mathrm{e}^{-2x}(c_2 - 4x) - 2x + 1/3.$$

2.1.3 Method of variation of parameters

Its applicability is wider than in the previous method, it can be used even when the coefficients of the equation are not constant, or when the independent term $R(x)$ is not of the any of the types previously considered.

Proposition 2.6.

A particular solution of the equation

$$y'' + P(x)\,y' + Q(x)\,y = R(x)$$

can be obtained with the expression

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)\,,$$

where y_1 and y_2 two linearly independent solutions of the associated homogeneous equation and the coefficients are

$$v_1(x) = \int \frac{-y_2(x)R(x)}{W(x)}\,dx\,, \qquad v_2(x) = \int \frac{y_1(x)R(x)}{W(x)}\,dx\,,$$

here W is the wronskian of y_1 and y_2. The resulting particular solution is

$$y_p(x) = \int_{x_0}^{x} \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W(s)} R(s)\,ds.$$

It is not necessary to include the integration constants when we compute v_1 and v_2 since we only need one of each.

***Example* 18.** The equation $y''+2y'+y = \mathrm{e}^{-x}\log x$ has $r = -1$ as the double root of the characteristic polynomial, this means that $y_1(x) = \mathrm{e}^{-x}$, $y_2(x) = x\mathrm{e}^{-x}$. The wronskian is $W(x) = \mathrm{e}^{-2x}$; and we look for a particular solution of the form $y_p(x) = v_1(x)\mathrm{e}^{-x} + v_2(x)x\mathrm{e}^{-x}$, where

$$v_1(x) = -\int \frac{x\mathrm{e}^{-x}\mathrm{e}^{-x}\log x}{\mathrm{e}^{-2x}}\,dx = x^2\left(\frac{1}{4} - \frac{1}{2}\log x\right),$$

$$v_2(x) = \int \frac{\mathrm{e}^{-x}\mathrm{e}^{-x}\log x}{\mathrm{e}^{-2x}}\,dx = x(\log x - 1).$$

Finally the particular solution we obtain is

$$y_p(x) = x^2\mathrm{e}^{-x}\left(\frac{1}{2}\log x - \frac{3}{4}\right).$$

2.2 Linear equations of order n

They have the general form:

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = R(x),$$

where $P_1, \ldots, P_n, R$ are functions of x. The equation is **homogeneous** when $R(x) = 0$. Since there are n derivatives, the space of solutions has dimension n. As with second order equations, the solution is

$$y(x) = y_h(x) + y_p(x),$$

where y_h is the general solution of the homogeneous equation (which contains n undetermined constants) and y_p is a particular solution of the full equation.

2.2.1 Constant coefficients

Let us consider

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = R(x), \qquad a_1, \ldots a_n \in \mathbb{R}.$$

The particular solutions of the equation can be found similarly as in second order. As for the homogeneous equation we look for solutions of the form $y = \mathrm{e}^{rx}$ as we did with order two, and obtain the **characteristic equation**:

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0.$$

1. **If there are n different roots:** $r_1, \ldots, r_n$, the solution is:

$$y_h(x) = c_1 \mathrm{e}^{r_1 x} + c_2 \mathrm{e}^{r_2 x} + \cdots c_n \mathrm{e}^{r_n x}.$$

2. **If one root has a higher multiplicity:** for example, if r_1 has multiplicity k, the associated solutions are:

$$(c_1 + c_2 x + \cdots + c_k x^{k-1})\mathrm{e}^{r_1 x}.$$

   We follow the same method with every multiple root.

3. **Complex conjugated roots**: If they are simple, $a \pm bi$ yield the solution:

$$\mathrm{e}^{ax}(A \cos bx + B \sin bx).$$

   If they have multiplicity k, they yield the solution:

$$\mathrm{e}^{ax}[(A_1 + A_2 x + \cdots + A_k x^{k-1}) \cos bx + (B_1 + B_2 x + \cdots + B_k x^{k-1}) \sin bx].$$

***Example* 19.** The equation $y^{(4)} - 2y''' + 2y'' - 2y' + y = 0$, has a characteristic equation with roots $r_1 = 1$ double, $r_{3,4} = \pm i$. The general solution is

$$y(x) = \mathrm{e}^x(c_1 + c_2 x) + c_3 \cos x + c_4 \sin x.$$

2.2.2 Equidimensional Euler equation

The general form of this kind of equations is:

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + a_2 x^{n-2} y^{(n-2)} + \cdots + a_n y = 0,$$

where we observe that all the terms have the same dimension. We look for solutions of the form $y = x^r$. Remember that if r is a complex number, $r = a \pm bi$, we have $y = x^a(c_1 \cos(b \log x) + c_2 \sin(b \log x))$. If some root r is multiple we seek for solutions of the form $y = x^r \log x$.

Alternative method: The coefficients of the equation become constant through the change of variable:

$$\left\{ \begin{array}{l} y(x) = z(t), \\ x = \mathrm{e}^t, \end{array} \right. \Longrightarrow t = \log x, \quad y'(x) = z'(t)\frac{1}{x}, \quad y''(x) = \frac{1}{x^2}(z''(t) - z'(t)) \, .$$

In general, the derivative $y^{(n)}$ always has the factor x^{-n}.

***Example* 20.** In the equation $x^3 y''' + 2x^2 y'' + xy' - y = 0$, writing $y = x^k$ we obtain $k(k-1)(k-2) + 2k(k-1) + k - 1 = 0$, with roots $k = 1$, $k = \pm i$. The general solution is

$$y(x) = c_1 x + c_2 \cos(\log x) + c_3 \sin(\log x).$$

With the change of variables the equation becomes $z''' - z'' + z' - z = 0$ and its general solution is i $z(t) = c_1 \mathrm{e}^t + c_2 \cos t + c_3 \sin t$. Undoing the change we obtain the same solution as before.

2.3 Applications

2.3.1 Electrical circuits

We have a generator, a resistance an alternator and a capacitor in a simple circuit:

Q	$\rightsquigarrow$	electric charge (coulombs)		
I	$\rightsquigarrow$	electric current (amps)	$\rightsquigarrow$	$I = Q'$
E	$\rightsquigarrow$	electromotive force (volts)		
R	$\rightsquigarrow$	resistance (ohms)	$\rightsquigarrow$	$E_R = RI$
L	$\rightsquigarrow$	inductance (henrys)	$\rightsquigarrow$	$E_L = LI'$
C	$\rightsquigarrow$	capacitance (farads)	$\rightsquigarrow$	$E_C = \dfrac{Q}{C}$.

Matching the electromotive force produced by the generator with the fall produced by the other three elements using **Kirchoff's law**, we have the equation

$$LI' + RI + \frac{Q}{C} = E$$

that is equivlent to the second order equations for the charge or the electric current, known as LRC equations,

$$LQ'' + RQ' + \frac{1}{C}Q = E, \qquad LI'' + RI' + \frac{1}{C}I = E'.$$

According to the kind of initial data, on the charge or on the current, we choose one or the other equation.

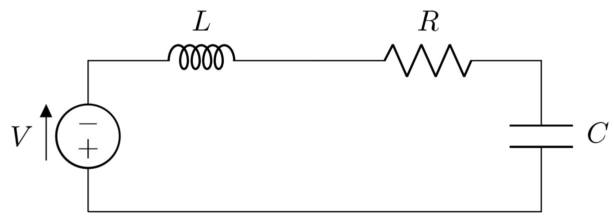

Figure 2.1: Basic electric circuit.

2.3.2 Mechanical systems

In the study of the forced vibrations of a body joined to a wall by a spring and with friction we consider:

x	$\rightsquigarrow$	displacement
m	$\rightsquigarrow$	mass
c	$\rightsquigarrow$	friction (force/velocity)
k	$\rightsquigarrow$	spring constant (force/distance)
F	$\rightsquigarrow$	external force.

Using **Newton's law** together with **Hooke's law** and the fact that the resistance due to friction is proportional to the velocity, we arrive to the equation

$$mx'' + cx' + kx = F.$$

The problem is complete when we fix the initial position and the initial velocity of the mass.

This model is very similar to the previous example, there is even a correspondence between the concepts:

$$x \leftrightarrow Q, \quad m \leftrightarrow L, \quad c \leftrightarrow R, \quad k \leftrightarrow \frac{1}{C}, \quad F \leftrightarrow E.$$

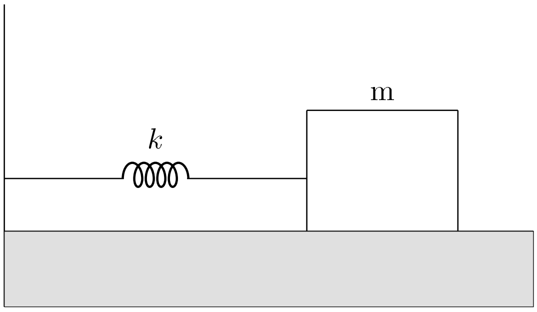

Figure 2.2: A mass attached to a spring, that slides on a table with friction.

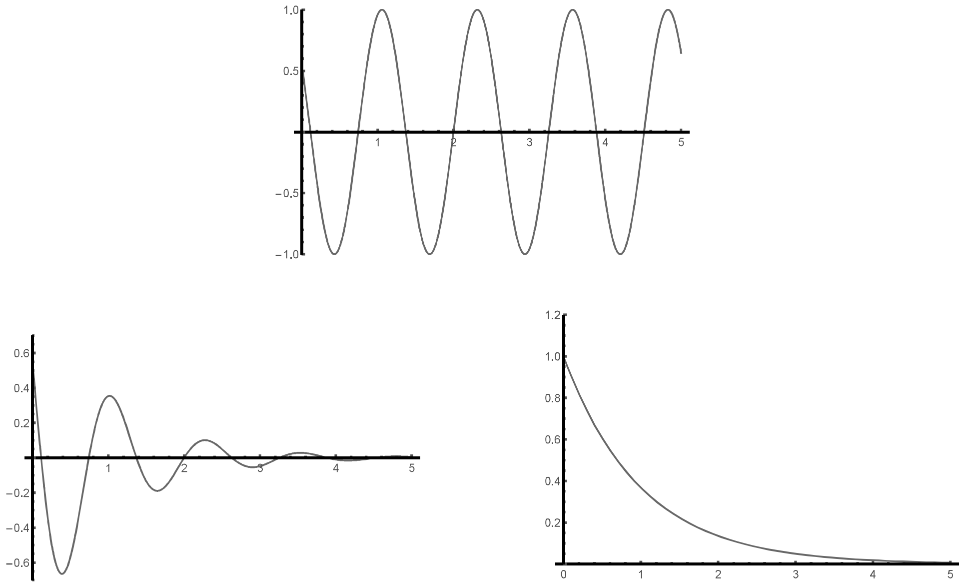

Figure 2.3: Examples of mechanical systems: oscillating ($c = 0$), under-deadening (c small), over-deadening (c big).

Depending on the relation between the friction and the constant of the spring multiplied by the mass, in particular of the sign of $c^2 - 4km$ (or analogously the resistance respect to L/C in the circuit) one obtains a system that is oscillating ($c = 0$), oscillating with deadening ($0 < c < 2\sqrt{km}$) or monotonous with over-deadening ($c \geq 2\sqrt{km}$).

– *ERC*–