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DIFFERENTIAL EQUATIONS

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Chapter 3

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3

LAPLACE TRANSFORM

The Laplace transformation is a tool that is useful in a very wide spectrum of problems in differential equations, integral equations and even linear systems of differential equations.

Contents

3.1 Definition and basic properties

We say that a function $f : (0,\infty) \longrightarrow \mathbb{R}$, **grows at most exponentially** if for some $\alpha \in \mathbb{R}$ it satisfies that:

$$\lim_{t\to\infty} f(t)\,\mathrm{e}^{-\alpha t} = 0.$$

In this case, if also f is integrable on each interval $(0,N)$, we can define the **Laplace transform** of f for $s > \alpha$, as the integral

$$F(s) = L[f](s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \lim_{N\to\infty} \int_0^N \mathrm{e}^{-st} f(t)\,dt\,.$$

Functions like polynomials, sine, cosine and e^{kx} grow at most exponentially, but for example e^{kx^2} does not. Along this chapter the functions to which we apply the Laplace transformation are always integrable and grow at most exponentially. We suppose also, when it is necessary, that they are zero on $(-\infty, 0)$. The Laplace transformation of a function exists even if the function is not defined at the origin (for example, we compute the Laplace transform of $f(x) = 1/\sqrt{x}$), but in most of the applications we need the value of $f(0)$.

Proposition 3.1.

Given f and g defined on $(0,\infty)$,

1. $L[\alpha f + \beta g](s) = \alpha\, L[f](s) + \beta\, L[g](s), \quad \alpha, \beta \in \mathbb{R}.$
2. $L[f(t)](s) \to 0$ *when* $s \to \infty$.
3. $L[\mathrm{e}^{at} f(t)](s) = L[f](s-a), \qquad a \in \mathbb{R}.$
4. $L[f(at)](s) = \dfrac{1}{a}\, L[f(t)]\left(\dfrac{s}{a}\right), \qquad a > 0.$

The derivatives are very easy to manage with this transformation, that is why this is a very useful tool in differential equations, there is one limitation: the problem must be defined on $(0,\infty)$, something that can be obtained easily in some cases with a shift.

Theorem 3.2.

1. *If f is derivable on $(0,\infty)$ and $f(0)$ exists, then*

$$L[f'](s) = s\,L[f](s) - f(0)\,.$$

2. *If f is twice derivable on $(0,\infty)$ and $f(0)$ and $f'(0)$ exist both, then*

$$L[f''](s) = s^2\, L[f](s) - s\, f(0) - f'(0)\,.$$

3. *$L[f]$ is derivable and*

$$\frac{d}{ds}\,(L[f](s)) = -L[t\, f(t)](s)\,.$$

4. *$L[f]$ has derivatives of any order and*

$$\frac{d^n}{ds^n}\,(L[f](s)) = (-1)^n L[t^n f(t)](s)\,.$$

Theorem 3.3.

$$L\left[\int_0^t f(x)\, dx\right](s) = \frac{1}{s}\, L[f](s)\,.$$

Theorem 3.4.

$$L\left[\frac{f(t)}{t}\right](s) = \int_s^\infty L[f(t)](u)\, du.$$

Most of the elementary Laplace transforms are obtained directly integrating or applying some of the properties we have described. It is also necessary the use of the **gamma function**, defined by:

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} dt, \quad x > 0.$$

Its basic properties are:

Proposition 3.5.

1. $\Gamma(1) = \Gamma(2) = 1.$
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$
3. $\Gamma(x+1) = x\Gamma(x), \quad x > 0.$
4. $\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$

3.1.1 List of Laplace transforms

$f(t)$	$L[f](s)$	
1	$\frac{1}{s}$,	$s > 0,$
t^n	$\frac{n!}{s^{n+1}}$,	$s > 0,\ n \in \mathbb{N},$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$,	$s > 0,\ a > -1,$
e^{at} ,	$\frac{1}{s-a}$,	$s > a,$
$\sin(at)$	$\frac{a}{s^2+a^2}$,	$s > 0,$
$\cos(at)$	$\frac{s}{s^2+a^2}$,	$s > 0,$
$\frac{\sin(at)}{t}$	$\arctan\frac{a}{s}$,	$s > 0,$
$\mathrm{e}^{at}t^b$	$\frac{\Gamma(b+1)}{(s-a)^{b+1}}$,	$s > a,\ b > -1,$
$\mathrm{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$,	$s > a,$
$\mathrm{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a,$
$\sinh(at) = \frac{\mathrm{e}^{at}-\mathrm{e}^{-at}}{2}$	$\frac{a}{s^2-a^2}$,	$s > a,$
$\cosh(at) = \frac{\mathrm{e}^{at}+\mathrm{e}^{-at}}{2}$,	$\frac{s}{s^2-a^2}$,	$s > a,$

3.2 Resolution of equations and linear systems

Transforming by Laplace a differential equation for the function $y(x)$, together with the initial data, it becomes an algebraic equation for the function $L[y](s)$, that we can solve. After that we undo the Laplace transformation with the **Laplace antitransformation**, that gives us back the solution if it is continuous, that is:

$$L[f](s) = L[g](s) \quad \Longleftrightarrow \quad f(x) = g(x),$$

if f and g are continuous. We do not use a formula, we are going to obtain it by decomposing the Laplace transform as a sum of simple fractions that we can identify as transforms of known functions, using the properties we have seen previously.

***Example* 21.**

$$\begin{cases} y'' + 4y = 4x \\ y(0) = 1,\ y'(0) = 5 \end{cases} \quad \Longrightarrow \quad s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{4}{s^2}\,.$$

where $Y = L[y]$. We clear $Y(s)$, decompose and antitransform:

$$Y(s) = \frac{4 + s^3 + 5s^2}{s^2(s^2+4)} = \frac{1}{s^2} + \frac{s+4}{s^2+4} \quad \Longrightarrow \quad y(x) = x + \cos 2x + 2\sin 2x.$$

The same method is used to solve **linear systems** of differential equations. The most simple case has two equations:

$$\begin{cases} x' = ax + by + c(t), \\ y' = dx + ey + f(t). \end{cases}$$

where $x = x(t)$ and $y = y(t)$ are the unknown functions , a, b, d, e are constants and $c(t), f(t)$ are functions of t. A linear system of two differential equations is equivalent to a linear differential equation of degree two and viceversa.

When we transform by Laplace a linear differential system we obtain a linear algebraic system in the variables $L[x](s)$ and $L[y](s)$, that can be solved using any algebraic method, like substitution, Gauss' method or Cramer's method. Finally we antitransform to find the solution $x(t), y(t)$.

***Example* 22.** The system

$$\begin{cases} x' = 4x - y, \\ y' = 2x + y, \\ x(0) = 0,\ y(0) = 1, \end{cases}$$

is transformed into (writing $X = L[x]$, $Y = L[y]$):

$$\begin{cases} sX = 4X - Y, \\ sY - 1 = 2X + Y. \end{cases}$$

Clearing for example X we obtain $X(s) = \dfrac{-1}{s^2 - 5s + 6}$, whose inverse transform is $x(t) = -e^{3t} + e^{2t}$. In order to obtain $y(t)$ we use the first equation of the original system, $y(t) = 4x(t) - x'(t) = -e^{3t} + 2e^{2t}$.

3.3 Advanced properties

3.3.1 Convolution

The **convolution** of two functions f and g is defined by:

$$f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

where f and g are regular enough so that the integral exists. Since the functions in which we are interested in this chapter are zero on $(-\infty, 0)$, we can write

$$f * g(x) = \int_0^x f(x-t)g(t)dt.$$

Theorem 3.6.

$$L[f * g](s) = L[f](s)\, L[g](s)\,.$$

Now we can transform many more expressions and also solve some **integral equations**, that are those equations in which some integral of the unknown function appear, or even **integro-differential equations**, where we have derivatives and also integrals of the unknown function.

***Example* 23.** **Volterra integral equation** has the form:

$$y(x) + \int_0^x K(x-t)y(t)dt = f(x), \quad x > 0,$$

where f and K are given functions, is solved as follows:

$$L[y](s) + L[K * y](s) = L[f](s) \quad \Longrightarrow \quad L[y](s) = \frac{L[f](s)}{1 + L[K](s)}.$$

Finally we antitransform.

3.3.2 Step function

When we deal with functions that are defined by pieces we use the **step function** or **Heaviside function**, defined by:

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0\,, \end{cases}$$

We use the Laplace transform of a function that is shifted and cut:

Theorem 3.7.

If $a \geq 0$ we have

$$L[f(t-a)H(t-a)](s) = \mathrm{e}^{-as} L[f(t)](s)\,.$$

In particular, $L[H(t-a)](s) = \mathrm{e}^{-as}/s$. To compute the Laplace transform of a function that is defined by pieces we first have to write it in terms of the function H:

***Example* 24.**

$$f(t) = \begin{cases} 3t, & 0 \leq t < 1 \\ t^2, & t \geq 1 \end{cases} = \begin{cases} 3t, & 0 \leq t < 1 \\ 3t + t^2 - 3t, & t \geq 1 \end{cases}$$

$$= \begin{cases} 3t, & 0 \leq t < 1 \\ 3t + (t-1)^2 - (t-1) - 2, & t \geq 1 \end{cases}$$

$$= 3t + H(t-1)\big((t-1)^2 - (t-1) - 2\big)\,.$$

$$L[f(t)](s) = \frac{3}{s^2} + \mathrm{e}^{-s}\left(\frac{2}{s^3} - \frac{1}{s^2} - \frac{2}{s}\right).$$

3.3.3 Dirac delta

We call **Dirac delta**, $\delta(t)$, to the limit:

$$\delta(t) = \lim_{\varepsilon \to 0} f_\varepsilon(t), \qquad \text{where} \quad f_\varepsilon(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \leq t \leq \varepsilon, \\ 0, & t > \varepsilon. \end{cases}$$

It is known as **impulse "function"** as an abuse of notation in some textbooks but a delta **is not** a function, that is why its properties are surprising:

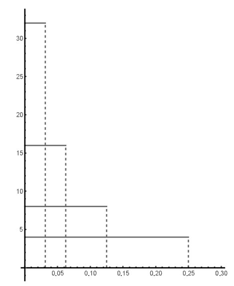

Figure 3.1: Approximation to the *Dirac delta*.

Proposition 3.8.

1. $\int_0^\infty \delta(t)dt = 1.$
2. $\int_0^\infty \delta(t-a)f(t)dx = f(a)$, *if* f *is continuous,* $a \geq 0$.
3. $H(x-a) = \int_0^x \delta(t-a)dt.$

Theorem 3.9.

If $a \geq 0$ *then*

$$L[\delta(t-a)](s) = \mathrm{e}^{-as}\,.$$

This result, or the previous property number 3, can be interpreted as if the Dirac delta is the derivative, in some sense, of the step function.

If in a differential equation there is a delta, it corresponds to the derivative of highest order, so that the equation can make sense. Then, if y'' has a delta then y' has a jump and y is continuous with a peak.

***Example* 25.**

$$\begin{cases} y'' + y = 1 + \delta(t-1), \\ y(0) = 1,\ y'(0) = 0. \end{cases} \implies (s^2+1)L[y](s) = s + \frac{1}{s} + \mathrm{e}^{-s}$$

$$\implies L[y](s) = \frac{1}{s} + \frac{\mathrm{e}^{-s}}{s^2+1}$$

$$\implies y(t) = 1 + \sin(t-1)H(t-1)$$

$$\implies y(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 1 + \sin(t-1), & t \geq 1. \end{cases}$$

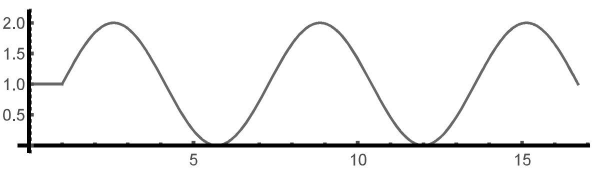

Figure 3.2: Solution of Example 25.

3.3.4 Summary of properties of the Laplace transform

The properties we have studied along this chapter, including the advanced properties, are summarized in the following table, where we call $F = L[f]$, $G = L[g]$, $W = L[w]$.

$f(t)$	$F(s)$
$f(t) = \mathrm{e}^{at}g(t)$	$G(s-a),\ s>a$
$f(t) = g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$
$f(t) = g'(t)$	$sG(s) - g(0)$
$f(t) = g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$f(t) = tg(t)$	$-G'(s)$
$f(t) = t^n g(t)$	$(-1)^n G^{n)}(s)$
$f(t) = \int_0^t g(x)\,dx$	$\frac{1}{s}G(s)$
$f(t) = \frac{g(t)}{t}$	$\int_s^\infty G(\tau)\,d\tau$
$f(t) = g * w(t)$	$G(s)W(s)$
$f(t) = H(t-a)g(t-a)$	$\mathrm{e}^{-as}G(s)$
$f(t) = \delta(t-a)$	e^{-as}

– *ERC*–