uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS

Degree in Biomedical Engineering

Chapter 4

Elena Romera

Open Course Ware, UC3M

http://ocw.uc3m.es/matematicas

4

METHOD OF SEPARATION OF VARIABLES

With this chapter we turn into partial differential equations, PDEs. As a necessary tool we study also Fourier series.

Contents

4.1 Introduction to partial differential equations

4.1.1 Definitions

A **partial differential equation (PDE)** is a relation of some partial derivatives of a several variables function $u : \mathbb{R}^N \to \mathbb{R}$, that is, an expression of the kind

$$F\left(\vec{x}, u(\vec{x}), \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}\right) = 0,$$

(we are going to study only equations of second order). The solution, if there is one, may contain arbitrary functions and we need data in order to obtain them.

***Example* 26.** For any function f:

$$u = yf(x) \quad \text{solves the equation} \quad y\frac{\partial u}{\partial y} = u.$$

In many examples one of the variables is time and the rest are spatial variables, $u : \Omega \times (0, \infty) \subset \mathbb{R}^{n+1} \to \mathbb{R}$, ($N = n + 1$ in this case). In this case we say that it is an evolution equation. The equations that do not depend on the time are usually called stationary $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $N = n$. A **problem in PDE** is an equation together with the data, that can be **initial contidions (IC)** if they are data for the time $t = 0$ in evolution equations, or **boundary conditions (BC)**, if they are data at the boundary of the spatial domain $\partial\Omega$. These last conditions can be of different kinds:

1. **Dirichlet conditions**: We know u at the boundary of the domain: $u = g$ on $\partial\Omega$, usually $g = 0$.

2. **Neumann conditions**: We know the normal derivative of u, that is the **flux**, on $\partial\Omega$, usually the flux is zero:

$$\frac{\partial u}{\partial \nu} = \vec{\nabla} u \cdot \vec{\nu} = g \quad \text{on} \quad \partial\Omega,$$

   where $\vec{\nu}$ is the outer normal vector to $\partial\Omega$.

3. **Mixed conditions:** u and some of its derivatives appear in the same boundary condition. For example

$$au + b\frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega.$$

The equations that we are going to study are **linear**, they can be written as:

$$Lu = f,$$

where L is a **linear operator**, that is, it satisfies:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v), \qquad \forall\, \alpha, \beta \quad \text{constants.}$$

This means that $L(u)$ is a linear combination of u and its derivatives:

$$L = \sum_{i=1}^{n} a_i(\vec{x}) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} b_{i,j}(\vec{x}) \frac{\partial^2}{\partial x_i \partial x_j}.$$

We have a **linear problem** if the equation and the boundary conditions are linear. We say that the PDE is **homogeneous** if $f = 0$, and $u \equiv 0$ is always a solution in these equations, it is known as **trivial solution.**

An **equilibrium solution** of a problem that depends on the time is a solution that is independent of the time. We expect usually that this is what we obtain when we compute the limit of the solution as $t \to \infty$.

4.1.2 Fundamental equations

The most important examples of operators in mathematical physics are:

$$L = \Delta, \qquad L = \frac{\partial}{\partial t} - \Delta, \qquad L = \frac{\partial^2}{\partial t^2} - \Delta,$$

that give rise, as we are going to see, to the Laplace equation, the heat equation and the wave equation. The first one is applied to functions $u : \mathbb{R}^n \to \mathbb{R}$, that is $u = u(\vec{x})$. The other two are applied to functions $u : \mathbb{R}^{n+1} \to \mathbb{R}$, that is $u = u(\vec{x}, t)$, with $\vec{x} \in \mathbb{R}^n$, $t > 0$, and are evolution operators, since as we have said, the variable t usually denotes the time. The **laplacian operator** appears in the three equations, this is the differential operator by excellence:

$$\Delta = \mathrm{div}\nabla = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad \big(= \nabla \cdot \nabla = \nabla^2 \quad \text{in physics notation}\big).$$

These three operators lead to the fundamental equations in PDE, because the second order linear equations can be classified according to the coefficients of second degree and the operator can be simplified into one of these three. In dimension two, for the equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = \text{ lower order terms}$$

there is always a simple change of variables that reduces the operator to one of them, depending on the sign of the discriminant $D = B^2 - 4AC$, the associated equation receives one of the following three names:

1. **Parabolic** if $D = 0$, with prototype the **heat equation**:

$$\frac{\partial u}{\partial t} = \Delta u.$$

This equation describes the distribution of temperatures $u(\vec{x}, t)$ in a body $\Omega \subset \mathbb{R}^n$ along time. the necessary physical data are initial distribution of temperatures and some boundary condition, for example zero temperature, or isolation, that can be described by:

$$\begin{array}{llll} \text{zero temperature} & \rightsquigarrow\ u = 0 & \text{on } \partial\Omega & \rightsquigarrow\ \text{Dirichlet cond.,} \\ \text{isolation} & \rightsquigarrow\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega & \rightsquigarrow\ \text{Neumann cond.} \end{array}$$

The condition $u = 0$ at the boundary can be obtained considering as the origin of temperatures a fixed temperature, that of the medium. The outer normal derivative to $\partial\Omega$, that is $\partial u/\partial \nu$, is the heat flux through and outside $\partial\Omega$; when it is zero there is no heat flux through the boundary and the body is isolated.

2. **Hiperbolic** if $D > 0$, and the prototype is the **wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

This equation describes the vibration of a string (in a violin, for example, in dimension $n = 1$), a membrane (a drum, in $n = 2$) or the propagation of waves in the space (acoustic or electromagnetic waves in $n = 3$). The more usual boundary conditions in the first two cases are Dirichlet conditions, that is, the string or the membrane are fixed at the boundary. We need two initial data, the initial position and also the initial velocity, because there is a second order derivative with respect to the time.

3. **Elliptic** if $D < 0$, with prototype the **Laplace equation**:

$$\Delta u = 0.$$

This equation describes the stationary distribution of temperatures or the equilibrium position of a vibrating string or membrane. Also it appears in many other physical or purely mathematical problems.

4.1.3 Simplifications

Linear problems can be simplified very much, so that in many cases it is only necessary to solve homogeneous problems, or with all but one of the boundary conditions homogeneous. Consider the problem:

$$(P) \quad \begin{cases} L(u) = f, & \text{on } \mathcal{D}, \\ B_j(u) = g_j, & \text{on } \Gamma_j,\ j = 1, \ldots, k, \end{cases}$$

where L is a linear operator, $u : \mathcal{D} \subset \mathbb{R}^N \to \mathbb{R}$ (where N can be n or $n+1$), and B_j for $j = 1, \ldots, k$ are linear boundary or initial conditions in a part of the boundary, $\Gamma_j \subset \partial\mathcal{D}$. Then:

1. If v solves the equation, then $w = u - v$ solves the new problem:

$$\begin{cases} L(w) = 0, & \text{on } \mathcal{D}, \\ B_j(w) = g_j - B_j(v), & \text{on } \Gamma_j,\ j = 1, \ldots, k. \end{cases}$$

2. If v satisfies the data, then $w = u - v$ solves the problem:

$$\begin{cases} L(w) = f - L(v), & \text{on } \mathcal{D}, \\ B_j(w) = 0, & \text{on } \Gamma_j,\ j = 1, \ldots, k. \end{cases}$$

3. **Superposition principle**: If $u_1, u_2, \ldots, u_m$, $m \in \mathbb{N}$, are solutions of the homogeneous problem:

$$\begin{cases} L(u) = 0, & \text{on } \mathcal{D}, \\ B_j(u) = 0, & \text{on } \Gamma_j,\ j = 1, \ldots, k, \end{cases}$$

   then, $u = \lambda_1 u_1 + \cdots + \lambda_m u_m$, with $\lambda_i \in \mathbb{R}$, is also a solution.

4. If u_0 solves the non homogeneous equation with homogeneous data and the functions u_j, $j = 1, \ldots, k$, solve the homogeneous equation with only the j-th datum non homogeneous, respectively:

$$\begin{cases} L(u_0) = f, & \text{on } \mathcal{D}, \\ B_j(u_0) = 0, & \text{on } \Gamma_j,\ j = 1, \ldots, k, \end{cases} \qquad \begin{cases} L(u_j) = 0, & \text{on } \mathcal{D}, \\ B_j(u_j) = g_j, & \text{on } \Gamma_j, \\ B_i(u_j) = 0, & \text{on } \Gamma_i,\ i \neq j, \end{cases}$$

   then the solution of problem (P) is

$$u = u_0 + u_1 + \cdots + u_k.$$

4.2 Method of separation of variables

4.2.1 General procedure

The method consists in looking for solutions in separate variable, that is, as a product of functions that depend each one on a different variable. In the heat equation:

$$\frac{\partial u}{\partial t} = \Delta u$$

we look for solutions of the form $u(\vec{x}, t) = \varphi(\vec{x})T(t)$, and arrive to

$$\varphi T' = T\Delta\varphi \implies \frac{T'}{T} = \frac{\Delta\varphi}{\varphi} = \text{constant},$$

because each side depends on a different variable. We denote the constant by $-\lambda$ since the sign simplifies future calculations.

We need as many data as derivatives in the equation and also we have to split the data into conditions for each one of those functions, so we prefer data with only one non homogeneous datum; if there are more than one we separate the problem into several problems, using the simplifications of the previous section, though this is not always possible. However, we can deal with several non homogenous data if they are initial data.

The spatial problem:

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0, \\ + \text{ boundary conditions}, \end{cases}$$

is an **eigenvalue problem**, since we can compute the values of λ (which are infinite as we are going to see), that we call **eigenvalues** and denote by λ_k, and the corresponding functions φ, that are known as **eigenfunctions**, φ_k. Once we know the eigenvalues we can solve the time equation,

$$T' + \lambda T = 0 \implies T(t) = \mathrm{e}^{-\lambda t},$$

and obtain a family of solutions in separate variables:

$$u_k(\vec{x}, t) = \mathrm{e}^{-\lambda_k t}\varphi_k(\vec{x}).$$

By the superposition principle, any linear combination of these functions is also a solution. So, if the initial codition is of the form $u(0, t) = \sum_{k=1}^{M} a_k\varphi_k(\vec{x})$, the solution has the expression

$$u(\vec{x}, t) = \sum_{k=1}^{M} a_k \mathrm{e}^{-\lambda_k t}\varphi_k(\vec{x}).$$

The question is then if any function can be written as a linear combination like that or under what conditions it is true. Let us see what happens in dimension $n = 1$.

4.2.2 Heat problem on a rod with zero temperature at finite ends

The problem is described by:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0, & \\ u(0,t) = 0,\ u(L,t) = 0, & t > 0, & \text{(Dirichlet BC)} \\ u(x,0) = f(x), & 0 < x < L. & \text{(IC)} \end{cases}$$

We look for product solutions of the form $u(x,t) = \varphi(x)T(t)$ that satisfy the equation and obtain

$$\frac{T'}{kT} = \frac{\varphi''}{\varphi} = -\lambda,$$

where λ is an arbitrary constant. We also separate the boundary conditions:

$$\begin{cases} u(0,t) = \varphi(0)T(t) = 0 \quad \forall t & \Longrightarrow \quad \varphi(0) = 0, \\ u(L,t) = \varphi(L)T(t) = 0 \quad \forall t & \Longrightarrow \quad \varphi(L) = 0, \end{cases}$$

because $T(t) \equiv 0$ implies $u(x,t) \equiv 0$, that is not a valid solution. Then, we have:

Time equation: $T' = -\lambda kT \implies T(t) = \mathrm{e}^{-k\lambda t}$.

Eigenvalue problem:

$$\begin{cases} \varphi'' + \lambda\varphi = 0, & 0 < x < L, \\ \varphi(0) = 0,\ \varphi(L) = 0. & \end{cases}$$

We are going to see that for some values of λ there exists a non trivial solution. Since the characteristic equation is $r^2 + \lambda = 0$ there are three different cases:

Case 1. $\lambda < 0 \implies r = \pm\sqrt{-\lambda}$, two different real roots, so for c_1, c_2 arbitrary constants:

$$\varphi(x) = c_1\mathrm{e}^{\sqrt{-\lambda}\,x} + c_2\mathrm{e}^{-\sqrt{-\lambda}\,x},$$
$$\varphi(0) = \varphi(L) = 0 \quad \Longrightarrow \quad c_1 = c_2 = 0 \quad \Longrightarrow \quad \varphi(x) \equiv 0.$$

Case 2. $\lambda = 0 \implies r = 0$, double root:

$$\varphi(x) = c_1 + c_2 x,$$
$$\varphi(0) = \varphi(L) = 0 \quad \Longrightarrow \quad c_1 = c_2 = 0 \quad \Longrightarrow \quad \varphi(x) \equiv 0.$$

Case 3. $\lambda > 0 \implies r = \pm i\sqrt{\lambda}$, pure imaginary roots:

$$\varphi(x) = c_1 \sin\left(\sqrt{\lambda}\, x\right) + c_2 \cos\left(\sqrt{\lambda}\, x\right)$$

$$\begin{cases} \varphi(0) = 0 \implies c_2 = 0, \\ \varphi(L) = 0 \implies c_1 \sin\left(\sqrt{\lambda}\, L\right) = 0 \implies \begin{cases} c_1 = 0 \implies \varphi(x) \equiv 0, \\ \sin\left(\sqrt{\lambda}\, L\right) = 0. \end{cases} \end{cases}$$

The last condition implies that $\sqrt{\lambda}\, L = n\pi,\ n = 1, 2, \ldots$ This gives us the eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

We have found the product solutions:

$$u_n(x,t) = \sin\frac{n\pi x}{L}\, \mathrm{e}^{-k(n\pi/L)^2 t}, \qquad n = 1, 2, \ldots$$

Using the superposition principle (linearity), any linear combination will also be a solution. In fact, if the series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\, \mathrm{e}^{-k(n\pi/L)^2 t}$$

converges "properly" then it will be also a solution of the heat equation with zero boundary conditions.

The initial condition is satisfied if we can find the coefficients b_n such that the initial condition can be written in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,.$$

This expression is a **Fourier sine series** and by the work of J. Fourier we know that almost "any" initial condition $f(x)$ can be written as a series like that. This is studied in the following section. Let us suppose by now that it is true. The b_n are called **Fourier coefficients** of $f(x)$ and we can compute them using the following **orthogonality relations**:

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq n, \\ L/2, & m = n, \end{cases} \qquad n, m \in \mathbb{N}.$$

If we fix $m \in \mathbb{N}$, multiply by $\sin\frac{m\pi x}{L}$ at both sides the expression of f and integrate on $[0, L]$, we obtain:

$$\int_0^L f(x) \sin\frac{m\pi x}{L}\, dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \frac{b_m L}{2}.$$

Then

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx.$$

Finally, the solution of our heat problem is:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}\,dy\right)e^{-k(n\pi/L)^2t}\sin\frac{n\pi x}{L} \\ &= \int_0^L f(y)\left(\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi y}{L}e^{-k(n\pi/L)^2t}\right)dy \\ &= \int_0^L f(y)G(x,y,t)\,dy. \end{aligned}$$

The function G is known as **Green's function** of the problem.

This method can also be applied with different boundary conditions: zero flux or mixed conditions. Also it can be applied in the wave equation and in Laplace's equation, even with more variables. The important question is to solve the eigenvalue problem and characterize the orthogonality relations.

4.3 Fourier series

4.3.1 Introduction

Now we study the series that we have found using the method of separation of variables. The series of the previous section is a particular case of other more general series that appear when we solve another PDEs problems. We start with a definition.

For a function f on the interval $[-L,L]$, its **Fourier series** is defined by:

$$S(f)(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L},$$

where the constants a_n, b_n are the **Fourier coefficients** and are defined by:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

These formulas make sense because the sine and cosine functions satisfy the following **orthogonality relations**:

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n, \end{cases} \qquad n, m \geq 1,$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n \geq 1, \\ 2L & m = n = 0, \end{cases} \qquad n, m \geq 0,$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = 0, \qquad n \geq 1,\ m \geq 0.$$

If the Fourier series of a function f exists on $[-L, L]$ that is, if it makes sense, the coefficients must be well defined, and this is obtained if

$$\int_{-L}^{L} |f(x)|\,dx < \infty.$$

The crucial questions now are if this series converges and if it converges to the function f. First of all, the Fourier series of f on $[-L, L]$ is periodic of period $2L$ but f is not necessarily periodic, so we consider the **periodic extension** of f with period $2L$, that we usually denote also by f:

$$f(x + 2KL) = f(x), \qquad \forall K \in \mathbb{Z}, \quad \forall x \in (-L, L].$$

The Fourier series of f can be different from the function and it is not clear if it is convergent or not, so the notation we use is:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

that means that the right hand side is the Fourier series of f. We have to define some concepts before we continue:

A function f is **piecewise C^1** or **piecewise regular** on $(-L, L)$ if the interval can be divided into subintervals such that f and f' are both continues on each open subinterval and the lateral limits exist and are finite at the endpoints, that is, f and f' can have **jump discontinuities** at the endpoints of the subintervals.

Theorem 4.1. ***(Convergence of Fourier series)***

If f is piecewise C^1 on $(-L, L)$, then the Fourier series of f converges to the periodic extension of f where the periodic extension is continuous and to the mean point of the two limits

$$\frac{1}{2}\big(f(x^+) + f(x^-)\big),$$

where the periodic extension has a jump, where

$$f(x^+) = \lim_{t \to x^+} f(t), \qquad f(x^-) = \lim_{t \to x^-} f(t)\,.$$

Special cases are the Fourier sine series and the Fourier cosine series, that we study now.

4.3.2 Fourier sine series

The symmetry may simplify the calculations in a Fourier series for some functions. If a function f is **odd** (that is, $f(-x) = -f(x)$) then $a_n = 0$ for every $n \geq 0$, so:

Proposition 4.2.

The Fourier series of an odd function is a sine series:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}\,,$$

where the coefficients are:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

When a function f is defined on $[0, L]$ and we need its Fourier sine series we define the **periodic odd extension** of f as the function of period $2L$ such that:

$$F(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in (-L, 0)\,, \end{cases}$$

and the Fourier series on $[-L, L]$ of this odd extension is the Fourier sine series of f on $[0, L]$.

***Example* 27.** The odd extension of $f(x) = x$ on $[0, L]$ to $[-L, L]$ is $F(x) = x$ and its Fourier sine series on $[0, L]$ is:

$$x \sim \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{L}.$$

4.3.3 Fourier cosine series

In an **even** function ($f(-x) = f(x)$) by the symmetry $b_n = 0$ for every $n \geq 1$, so:

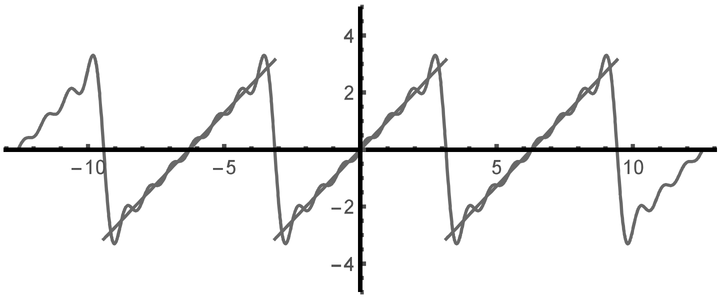

Figure 4.1: Sum of the first seven terms of the Fourier sine series of $f(x) = x$ on $[-\pi, \pi]$, compared with the odd periodic extension.

PROPOSITION 4.3.

The Fourier series of an even function is a cosine series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

with coefficients:

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = \frac{1}{L} \int_{0}^{L} f(x)\, dx.$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx.$$

Given a function f on $[0, L]$, when we need its Fourier cosine series we define the **periodic even extension** of f as the function of period $2L$ such that:

$$F(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in (-L, 0), \end{cases}$$

and the Fourier series of this extension to $[-L, L]$ is the Fourier cosine series of f on $[0, L]$.

***Example* 28.** The even extension of $f(x) = x$ on $[0, L]$ to $[-L, L]$ is $F(x) = |x|$ and its Fourier cosine series on $[0, L]$ is:

$$x \sim \frac{L}{2} + \sum_{k=0}^{\infty} \frac{-4L}{(2k+1)^2 \pi^2} \cos \frac{(2k+1)\pi x}{L}.$$

In the figures of the Fourier sine and cosine series of the function $f(x) = x$ on $[-\pi, \pi]$ we observe that the second converges more quickly.

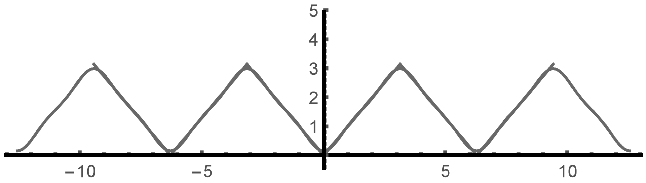

Figure 4.2: Sum of the first three terms of the Fourier cosine series of $f(x) = x$ on $[-\pi, \pi]$, compared with the periodic even extension.

4.3.4 Continuity of Fourier series

As a consequence of the previous definitions and the convergence of the Fourier series we have:

Theorem 4.4. ***(Continuity)***

1. ***General Fourier series***: *If f is piecewise C^1 on $(-L, L)$, then its Fourier series is continuous if and only if f is continuous on $[-L, L]$ and $f(-L) = f(L)$.*

2. ***Cosine series***: *If f is piecewise C^1 on $(0, L)$, then its Fourier cosine series is continuous if and only if f is continuous on $[0, L]$.*

3. ***Sine series***: *If f is piecewise C^1 on $(0, L)$, then its Fourier sine series is continuous if and only if f is continuous on $[0, L]$ and $f(0) = f(L) = 0$.*

Observe that the series that needs more conditions is the sine series and the series that requires less conditions is the cosine series.

4.3.5 Other properties of the Fourier series

When we use Fourier series to solve PDEs it is necessary to derive and to integrate this kind of expressions. Now we study the conditions with which this can be done.

Theorem 4.5. ***(Derivability)***

1. ***General Fourier series***: *If the Fourier series of f is continuous and piecewise C^1, then it can be derived term by term, and the series we obtain is the Fourier series of f' (that converges to f' at the points of continuity of f').*

2. ***Fourier cosine series***: *If f is continuous on $[0, L]$ and piecewise C^1 on $(0, L)$, then its Fourier cosine series can be derived term by term, and the series we obtain is the Fourier sine series of f' (that converges to f' at the points of continuity of f').*

3. ***Fourier sine series***: *If f is continuous on $[0, L]$ and piecewise C^1 on $(0, L)$, then its Fourier sine series can be derived term by term if and only if $f(0) = f(L) = 0$. In case we can derive term by term, the series we obtain is the Fourier cosine series of f' (that converges to f' at the points of continuity of f').*

Observe that, again, the sine series are more demanding. Nevertheless, if we have a sine series of a function f that is continuous on $[0, L]$ and piecewise C^1 on $(0, L)$,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \,,$$

it can be proved that the Fourier cosine series of the derivative is:

$$f'(x) \sim \frac{1}{L}\big(f(L) - f(0)\big) + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n + \frac{2}{L}\big((-1)^n f(L) - f(0)\big) \right) \cos \frac{n\pi x}{L} \,,$$

and, as can be seen directly, this coincides with the derivative term by term of the Fourier sine series of f if and only if $f(0) = f(L) = 0$.

Sometimes we need the derivative with respect to variables that do not appear in the eigenfunctions, and this is easy:

Theorem 4.6. ***(Derivative with respect to a parameter)***

If $u = u(x, t)$ is a continuous function on $[-L, L] \times [0, \infty)$ and $\partial u / \partial t$ is piecewise C^1 as a function of $x \in (-L, L)$ for every $t \in [0, \infty)$, then its Fourier series

$$u(x, t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L} \,,$$

can be derived term by term with respect to the parameter t, and we obtain

$$\frac{\partial u}{\partial t}(x, t) \sim a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{L} \,.$$

With respect to integration we have:

Theorem 4.7. ***(Integration)***

If f is piecewise C^1, then the three Fourier series can be integrated term by term, and the result is a series that is convergent for all $x \in [-L, L]$ to the integral of f.

However, it may happen that the series we obtain is **not** a Fourier series, as we can see:

If f is piecewise C^1 on $(-L, L)$ and has a Fourier series on that interval:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},$$

term by term integration gives:

$$\int_{-L}^{x} f(t)\, dt \sim a_0(x+L) + \sum_{n=1}^{\infty}\left(\frac{a_n L}{n\pi}\sin\frac{n\pi x}{L} + \frac{b_n L}{n\pi}\left((-1)^n - \cos\frac{n\pi x}{L}\right)\right),$$

that, as can be seen easily, is a Fourier series if and only if $a_0 = 0$.

4.4 More examples of separation of variables

4.4.1 Heat equation on a rod with isolated endpoints

The problem is now

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, & t > 0, \qquad \text{(Neuman BC)}, \\ u(x,0) = f(x), & 0 < x < L, \qquad \text{(IC)}. \end{cases}$$

We look for solutions in the form $u(x,t) = \varphi(x)T(t)$. The time equation is the same as in the previous example on the rod (subsection 4.2.2) and the eigenvalue problem for x is:

$$\begin{cases} \varphi''(x) + \lambda\phi(x) = 0 \quad \Longrightarrow \quad \varphi(x) = \mathrm{e}^{rx}, \quad r^2 + \lambda = 0. \\ \varphi'(0) = \varphi'(L) = 0. \end{cases}$$

The eigenvalues and eigenfunctions are now:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = c_1 \cos\frac{n\pi x}{L}, \quad n = 0, 1,\ 2,\ 3,\ \ldots$$

where $\varphi_0(x) = 1$. Then, using the superposition principle we obtain the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, \mathrm{e}^{-(n\pi/L)^2 kt},$$

The initial condition implies that

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

This is a Fourier cosine series. The **orthogonality relations** for the eigenfunctions are:

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \neq 0 \\ L, & n = m = 0 \end{cases} \qquad n, m \in \mathbb{N}.$$

and with them we obtain:

$$A_0 = \frac{1}{L}\int_0^L f(y)\, dy, \qquad A_n = \frac{2}{L}\int_0^L f(y) \cos \frac{n\pi y}{L}\, dy, \quad n = 1, 2, \ldots$$

Joining everything we arrive to the representation of the solution by the Green's function

$$u(x,t) = \int_0^L f(y) G(x,y,t)\, dy,$$

where

$$G(x,y,t) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \cos \frac{n\pi y}{L}\, \mathrm{e}^{-(n\pi/L)^2 kt}.$$

4.4.2 Laplace equation on a disc

On the disc of center $(0,0)$ and radius a we use polar coordinates, $u = u(r,\theta)$ and the laplacian becomes:

$$\begin{cases} \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, & -\pi < \theta < \pi, \qquad r > 0, \\ u(a,\theta) = f(\theta), & r > 0, \qquad \text{(BC)}. \end{cases}$$

The interval $[-\pi, \pi]$ is the best option for further calculations.

There are four derivatives and only one condition, so we need three more conditions. These are known as **implicit conditions** (that are not given directly in the problem):

1. **Boundedness at the origin** (or non singularity): $|u(0,\theta)| < \infty$.

2. **Periodicity conditions**:

$$u(r,-\pi) = u(r,\pi), \qquad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi).$$

Now we look for product solutions $u(r,\theta) = \varphi(\theta)G(r)$. The periodicity conditions give rise to two conditions for φ and we arrive to the following eigenvalue problem that is the **angular problem**:

$$\begin{cases} \varphi'' + \lambda\varphi = 0, \\ \varphi(-\pi) = \varphi(\pi), \\ \varphi'(-\pi) = \varphi'(\pi). \end{cases}$$

The eigenvalues are $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ and the corresponding eigenfunctions:

$$\sin n\theta, \quad \text{and} \quad \cos n\theta, \qquad n = 0, 1, 2, \ldots$$

For $n = 0$ observe that there is only one eigenfunction, $\varphi_0(\theta) = 1$. For the rest of the eigenvalues there are two eigenfunctions associated.

The **radial problem** has an Euler type equation:

$$\begin{cases} r^2G'' + rG' - n^2G = 0, \\ \\ |G(0)| < \infty, \end{cases} \implies \begin{cases} G_n(r) = c_1r^n + c_2r^{-n}, & n > 0, \\ G_0(r) = k_1 + k_2\log r, & n = 0, \end{cases}$$

With the boundedness condition, $c_2 = k_2 = 0$. Using the superposition principle we obtain the solution:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_nr^n\cos n\theta + B_nr^n\sin n\theta\right).$$

If f can be written as a general Fourier series:

$$u(a,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty}\left(A_na^n\cos n\theta + B_na^n\sin n\theta\right),$$

we can obtain the coefficients using the orthogonality relations of sines and cosines of the Fourier series (in the previous section). In our case, $L = \pi$ and the coefficients are:

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi, \\ (n \geq 1) \quad A_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\cos n\phi\, d\phi, \\ B_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\sin n\phi\, d\phi. \end{aligned}$$

Again we can write the solution using a Green's function

$$u(r,\theta) = \int_{-\pi}^{\pi} f(\phi)G(r,\theta,\phi)\, d\phi,$$

where

$$G(r,\theta,\phi)=\frac{1}{2\pi}+\frac{1}{\pi a^n}\sum_{n=1}^{\infty}\left(\cos n\theta\cos n\phi+\sin n\theta\sin n\phi\right)r^n.$$

– $\mathcal{ERC}$–