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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS

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Chapter 5

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STURM-LIOUVILLE PROBLEMS

In this chapter we extend the resolution of PDEs using separation of variables to more general cases of eigenvalue problems.

Contents

5.1 Introduction

The resolution of PDEs with the method of separation of variables implies the solution of an eigenvalue problem. The eigenvalue problems that appear in the previous chapter are particular cases of much more general cases that we study now.

5.1.1 Examples

In the following cases we see how, in a natural way, appear some eigenvalue problems that we still haven't solved.

1. **Heat equation on a non-uniform rod with heat sources**:

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) + Q,$$

where $c(x)$ is the specific heat, $\rho(x)$ the density, $K(x)$ the conductivity and $Q(x,t)$ the heat sources, that we suppose are of the form $Q(x,t) = \alpha(x)u(x,t)$. With homogeneous boundary conditions we can separate variables, $u(x,t) = \varphi(x)G(t)$,

$$c\rho\varphi G' = G\left(K\varphi'\right)' + \alpha\varphi G,$$

and arrive to the following spatial equation:

$$\left(K\varphi'\right)' + \alpha\varphi + \lambda c\rho\varphi = 0.$$

2. **Vibration of a non-uniform string with friction**:

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2} + \alpha\varphi,$$

Here $\rho(x)$ is the density, T the tension, that we suppose constant, and $\alpha(x)$ represents the friction. The spatial equation obtained when we separate variables with homogeneous boundary conditions is:

$$T\varphi'' + \alpha\varphi + \lambda\rho\varphi = 0.$$

The two equations we have obtained are of the same form, that we study now in general.

5.1.2 Definitions and fundamental theorem

A **regular eigenvalue Sturm-Liouville problem** consists in a **Sturm-Liouville equation**, that is of the form:

$$\left(p\varphi'\right)' + q\varphi + \lambda\sigma\varphi = 0, \qquad a < x < b,$$

where p, q and σ are functions of x, together with **regular** boundary conditions, that are of the form:

$$\begin{aligned} \beta_1\varphi(a) + \beta_2\varphi'(a) &= 0, \\ \beta_3\varphi(b) + \beta_4\varphi'(b) &= 0, \end{aligned}$$

here $\beta_i \in \mathbb{R}$, $i = 1, \cdots, 4$. Besides, the functions p, q and σ must be continuous on the interval $[a, b]$, with $p > 0$ and $\sigma > 0$. The unknowns are the **eigenfunction** φ and the **eigenvalue** λ.

The regular boundary conditions include Dirichlet and Newmann conditions, but not the boundedness condition nor the periodicity conditions.

Even when we are not able to solve these problems, we know many properties of them, as we see in the following theorem.

Theorem 5.1. ***(Sturm-Liouville)*.**

Every regular Sturm-Liouville problem satisfies that:

1. *All the eigenvalues λ are real and form an infinite sequence*

   $$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$

   with a minimum eigenvalue, usually denoted by λ_1 and with $\lambda_n \to \infty$ when $n \to \infty$.

2. *To each eigenvalue λ_n corresponds one eigenfunction, denoted by φ_n (unique but for multiplicative constants), that has exactly $n - 1$ zeros on the interval $a < x < b$.*

3. *The eigenfunctions φ_n form a "complete" set, that is, any piecewise regular functions f can be represented as a **generalized Fourier series** of the eigenfunctions:*

   $$f(x) \sim \sum_{n=1}^{\infty} a_n\varphi_n(x).$$

   Even more, this series converges to the value $\dfrac{f(x^+) + f(x^-)}{2}$ on $a < x < b$ (if the coefficients a_n are chosen properly).

4. *The eigenfunctions that correspond to different eigenvalues are orthogonal with respect to the weight function σ. In other words,*

$$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)\, dx = 0, \quad \text{if } \lambda_n \neq \lambda_m.$$

5. *Each eigenvalue is related with its eigenfunction by the* ***Rayleigh quotient****:*

$$\lambda = RQ(\varphi) = \frac{\Big[-p\varphi\,\varphi' \Big]_a^b + \displaystyle\int_a^b \left(p\,(\varphi')^2 - q\varphi^2 \right)}{\displaystyle\int_a^b \varphi^2\sigma}.$$

The boundary conditions can simplify this formula (the first term above is zero in many cases). Besides, the theorem is valid almost completely also for Sturm-Liouville eigenvalue problems with boundary conditions that are not regular, such as periodicity conditions or boundedness conditions. In these problems, those conditions are slightly different from the ones we have seen in the previous chapter:

Periodicity conditions:

$$\begin{cases} \varphi(a) = \varphi(b), \\ p(a)\varphi'(a) = p(b)\varphi'(b). \end{cases}$$

Boundedness condition: $|\varphi(x)| < \infty$ when $p(x) = 0$, so that the product $p(x)\varphi(x)$ is zero at those points.

With periodicity and boundedness conditions we have seen examples in which there is no uniqueness for the eigenfunctions, nevertheless, the rest of the theorem remains true.

5.1.3 Illustration of the theorem

The simplest example of Sturm-Liouville regular problem is:

$$\begin{cases} \varphi'' + \lambda\varphi = 0, \\ \varphi(0) = \varphi(L) = 0. \end{cases}$$

As we know, the eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots,$$

Now we check the theorem:

1. **Eigenvalues**: There are infinite eigenvalues and they are real: $\lambda_n = (n\pi/L)^2$, with $n = 1, 2, 3, \ldots$, with a minimum eigenvalue, $\lambda_1 = (\pi/L)^2$, and $\lambda_n \to \infty$ as $n \to \infty$.

2. **Eigenfunctions**: To each eigenvalue corresponds a unique eigenfunction (except for the product by constants) $\varphi_n(x) = \sin n\pi x/L$. The n-th eigenfunction has exactly $n-1$ zeros on $(0, L)$: $x_j = jL/n$, $j = 1, \cdots, n-1$.

3. **Eigenfunctions series**: We can represent any piecewise regular function f as a series of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L},$$

that is the Fourier sine series of f and we know that converges to $[f(x^+) + f(x^-)]/2$ for $0 < x < L$, so it converges to $f(x)$ for $0 < x < L$ if it is a continuous function.

4. **Orthogonality of the eigenfunctions**: Since the weight is $\sigma(x) = 1$, the orthogonality condition reduces to

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0, \qquad n \neq m.$$

This is the orthogonality that allows us to calculate the coefficients of the series.

5. **Rayleigh quotient**: We check that it gives the desired value of the eigenvalue associated to each eigenfunction:

$$\begin{aligned}
\lambda_n &= \frac{\Big[-p\varphi_n\varphi_n'\Big]_a^b + \displaystyle\int_a^b \Big(p(\varphi_n')^2 - q\varphi_n^2\Big)}{\displaystyle\int_a^b \varphi_n^2 \sigma} = \frac{\displaystyle\int_0^L (\varphi_n')^2}{\displaystyle\int_0^L \varphi_n^2} \\
&= \frac{\displaystyle\int_0^L \Big(\frac{n\pi}{L}\Big)^2 \cos^2 \frac{n\pi x}{L}}{\displaystyle\int_0^L \sin^2 \frac{n\pi x}{L}} = \frac{(n\pi/L)^2 L/2}{L/2} = \Big(\frac{n\pi}{L}\Big)^2.
\end{aligned}$$

5.2 Generalized Fourier series

The coefficients of the generalized Fourier series can be obtained applying the orthogonality of the eigenfunctions with respect to the weight σ, following part four of the theorem. The procedure is similar to that used with Fourier series. We start with a series:

$$f(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x), \qquad x \in [a, b],$$

multiply by $\varphi_m(x)$ (with fixed m) and $\sigma(x)$ so that we can use the orthogonality. We suppose that all the calculations that we need to perform with the series are valid, so we use the equality sign,

$$f(x)\varphi_m(x)\sigma(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)\varphi_m(x)\sigma(x), \qquad x \in [a, b].$$

Integrating on $[a, b]$ we obtain

$$\int_a^b f(x)\varphi_m(x)\sigma(x)\ dx = \sum_{n=1}^{\infty} a_n \int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)\ dx.$$

By the orthogonality with respect to the weight σ all the integrals in the right hand side are zero except for the corresponding to $n = m$:

$$\int_a^b f(x)\varphi_m(x)\sigma(x)\ dx = a_m \int_a^b \varphi_m^2(x)\sigma(x)\ dx.$$

The right hand side integral is not null because $\sigma > 0$ by definition of regular Sturm-Liouville problem and φ_n cannot be identically zero because it is an eigenfunction. Then we can divide by this integral to compute the generalized Fourier coefficient a_m:

$$a_m = \frac{\displaystyle\int_a^b f(x)\varphi_m(x)\sigma(x)\ dx}{\displaystyle\int_a^b \varphi_m^2(x)\sigma(x)\ dx}.$$

5.3 Rayleigh quotient and Minimization Principle

The Rayleigh quotient can be obtained from the Sturm-Liouville differential equation:

$$(p\varphi')' + q\varphi + \lambda\sigma\varphi = 0.$$

Multiplying by φ and integrating we arrive to

$$\int_a^b \left(\varphi\,(p\varphi')' + q\varphi^2\right) + \lambda \int_a^b \varphi^2\sigma = 0,$$

and since $\int_a^b \varphi^2\sigma > 0$, we can clear λ:

$$\lambda = \frac{-\int_a^b \left(\varphi\left(p\varphi'\right)' + q\varphi^2\right)}{\int_a^b \varphi^2\sigma}.$$

Finally we integrate by parts,

$$\lambda = \frac{\Big[-p\varphi\varphi'\Big]_a^b + \int_a^b \left(p\left(\varphi'\right)^2 - q\varphi^2\right)}{\int_a^b \varphi^2\sigma}.$$

The Rayleigh quotient gives us a very useful information about the eigenvalues (for example, the sign) precisely when the problem cannot be solved. Also we can estimate the first eigenvalue with the following:

Theorem 5.2. ***(Minimization principle)***

The first eigenvalue of a regular Sturm-Liuville problem fulfills:

$$\lambda_1 = \min_u \frac{\left[-puu'\right]_a^b + \int_a^b \left(p\left(u'\right)^2 - qu^2\right)}{\int_a^b u^2\sigma},$$

where the minimum is calculated over all the piecewise continuous functions on (a, b) that satisfy the boundary conditions.

The minimum value is attained only for $u = \varphi_1$, the eigenfunction that corresponds to the minimum eigenvalue. In problems, for example, with the heat equation, the minimum eigenvalue has a great importance because it usually represents the large time behaviour of the solution.

5.4 Bessel equation

5.4.1 Wave equation on a disc: radial case

We use polar coordinates to describe the problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right), & 0 < r < a, \quad t > 0, \\ u(a, t) = 0, & t > 0, \\ u(r, 0) = \alpha(r), & 0 < r < a, \\ u_t(r, 0) = \beta(r), & 0 < r < a. \end{cases}$$

Since we suppose that the solution is radial there is no dependance on the angle in the laplacian. We also need the implicit condition of boundedness at the origin:

$$|u(0,t)| < \infty.$$

We separate $u(r,t) = \varphi(r)G(t)$ and obtain the time equation:

$$G''(t) + \lambda c^2 G(t) = 0,$$

and the spatial problem:

$$\begin{cases} (r\varphi')' + \lambda r\varphi = 0, \quad 0 < r < a, \\ \varphi(a) = 0, \\ |\varphi(0)| < \infty. \end{cases}$$

This is not a regular Sturm-Liouville problem, ($p(r) = r$, $q(r) = 0$, $\sigma(r) = r$) but we can still compute the Rayleigh quotient and obtain that:

$$\lambda = \frac{[-r\varphi(r)\varphi'(r)]_0^a + \displaystyle\int_0^a r(\varphi')^2(r)\,dr}{\displaystyle\int_0^a \varphi^2(r)\,r\,dr} = \frac{\displaystyle\int_0^a r(\varphi')(r)^2\,dr}{\displaystyle\int_0^a \varphi^2(r)\,r\,dr} \geq 0,$$

Also, $\lambda = 0 \implies \varphi'(r) = 0 \implies \varphi(r) = Constant = 0$, so $\lambda = 0$ is not an eigenvalue and so all the eigenvlues are positive. Then we can perform the change of variables:

$$z = \sqrt{\lambda} r, \qquad \frac{d}{dr} = \sqrt{\lambda}\frac{d}{dz}, \qquad \varphi(r) = \psi(z),$$

and if we multiply also by z we arrive to:

$$z^2\psi'' + z\psi' + z^2\psi = 0.$$

In the following subsection we study this equation.

5.4.2 Bessel equations and functions

We call **Bessel equation of order zero** to:

$$z^2\psi'' + z\psi' + z^2\psi = 0,$$

and the **Bessel equation of order m** is:

$$z^2\psi'' + z\psi' + (z^2 - m^2)\psi = 0.$$

They have the solutions $\psi(z) = c_1 J_m(z) + c_2 Y_m(z)$, where J_m and Y_m are respectively the **first and second kind Bessel functions of order m**. They are oscillating functions, with the following behaviour as they approach $z \to 0$:

$$J_0(z) \sim 1, \qquad Y_0(z) \sim \log z,$$

$$J_m(z) \sim z^m, \qquad Y_m(z) \sim z^{-m}, \quad m \geq 1.$$

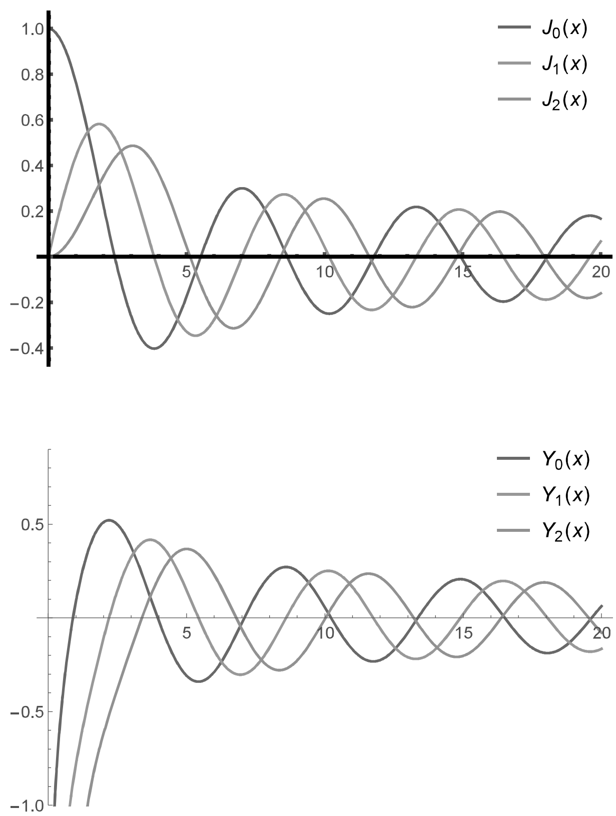

Figure 5.1: Bessel functions of first and second kind.

Thus, the first kind functions are bounded at the origin, while the functions of second kind are not. Also, the zeros of these functions form an increasing sequence, and though they cannot be calculated in an exact form, they are tabulated and are known approximately. Figure 5.1 shows the graphs of the first three Bessel functions of first and second kind.

5.4.3 Resolution of the radial wave equation with Bessel functions

We continue the resolution of the radial wave equation on a disc, now we undo the change and get:

$$\varphi(r) = c_1 J_0(r\sqrt{\lambda}) + c_2 Y_0(r\sqrt{\lambda})$$

The boundedness at the origin implies $c_2 = 0$ and the boundary condition gives:

$$J_0(a\sqrt{\lambda}) = 0, \quad \Longrightarrow \quad a\lambda = \eta,$$

where η is one of the infinite zeros of J_0: $\{\eta_{0,n}\}_{n=1}^{\infty}$. We have found the eigenvalues:

$$\lambda_n = \left(\frac{\eta_{0,n}}{a}\right)^2, \quad n = 1, 2, \ldots$$

The eigenfunctions are then:

$$\varphi_n(r) = J_0\left(\frac{\eta_{0,n} r}{a}\right), \quad n = 1, 2, \ldots$$

The time equation can be solved now using that the eigenvalues are positive:

$$G_n(t) = c_1 \cos\left(\frac{c\eta_{0,n} t}{a}\right) + c_2 \sin\left(\frac{c\eta_{0,n} t}{a}\right), \quad n = 1, 2, \ldots$$

Finally, the solution of our problem has the form:

$$u(r,t) = \sum_{n=1}^{\infty} J_0\left(\frac{\eta_{0,n} r}{a}\right)\left\{A_n \cos\left(\frac{c\eta_{0,n} t}{a}\right) + B_n \sin\left(\frac{c\eta_{0,n} t}{a}\right)\right\}.$$

The eigenfunctions are solutions of a Sturm-Liouville problem, so they are orthogonal with respect to the weight $\sigma(r) = r$ and form a complete system. This allows us to calculate the coefficients:

$$u(r,0) = \alpha(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\eta_{0,n} r}{a}\right) \Longrightarrow A_n = \frac{\displaystyle\int_0^a \alpha(r) J_0\left(\frac{\eta_{0,n} r}{a}\right) r\, dr}{\displaystyle\int_0^a J_0^2\left(\frac{\eta_{0,n} r}{a}\right) r\, dr},$$

$$u_t(r,0) = \beta(r) = \sum_{n=1}^{\infty} B_n \frac{c\eta_{0,n}}{a} J_0\left(\frac{\eta_{0,n} r}{a}\right) \Longrightarrow B_n = \frac{\displaystyle\int_0^a \beta(r) J_0\left(\frac{\eta_{0,n} r}{a}\right) r\, dr}{\displaystyle\frac{c\eta_{0,n}}{a}\int_0^a J_0^2\left(\frac{\eta_{0,n} r}{a}\right) r\, dr}.$$

The representation of the solution using Green's function is more complicated in this case.

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