uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS

Degree in Biomedical Engineering

Chapter 6

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FOURIER TRANSFORM

We dedicate this chapter to a very powerful mathematical tool: the Fourier transform and its application to the resolution of partial differential equations.

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6.1 Fourier transform on $\mathbb{R}$

6.1.1 Definition and basic properties

If f is integrable on $\mathbb{R}$ we define its **Fourier transform** as:

$$\widehat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{i\omega x}\, dx\,.$$

This transformation is related with the Laplace transform, that was defined only for positive values. The following is a formula to recover the function from its Fourier transform, it is called **inverse Fourier transform** and it is denoted by $\mathcal{F}^{-1}$:

Theorem 6.1.

If f is continuous then:

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x) = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, \mathrm{e}^{-i\omega x}\, d\omega.$$

The main properties of the Fourier transform are:

Theorem 6.2.

1. $\mathcal{F}(af + bg)(\omega) = a\widehat{f}(\omega) + b\widehat{g}(\omega), \quad a, b \in \mathbb{R}.$
2. $\mathcal{F}(f(x - x_0))(\omega) = \mathrm{e}^{i\omega x_0}\widehat{f}(\omega), \quad x_0 \in \mathbb{R}.$
3. $\mathcal{F}(f(\alpha x))(\omega) = \frac{1}{|\alpha|}\,\widehat{f}\left(\frac{\omega}{\alpha}\right), \quad \alpha \in \mathbb{R} \setminus \{0\}.$
4. $\mathcal{F}\left(f'\right)(\omega) = -i\omega\widehat{f}(\omega).$
5. $\mathcal{F}\Big(\frac{d^k f}{dx^k}\Big)(\omega) = (-i\omega)^k\widehat{f}(\omega), \quad k \in \mathbb{N}.$
6. *If $f = f(x,t)$, then* $\mathcal{F}\left(\frac{\partial f}{\partial t}\right)(\omega) = \frac{\partial \widehat{f}}{\partial t}(\omega).$

***Example* 29.** Some important transformations are: (for $\alpha > 0$).

1. $\mathcal{F}\big(\mathrm{e}^{-\alpha x^2}\big)(\omega) = \frac{1}{\sqrt{4\pi\alpha}}\,\mathrm{e}^{-\frac{\omega^2}{4\alpha}}, \quad \mathcal{F}\Big(\sqrt{\frac{\pi}{\alpha}}\,\mathrm{e}^{-\frac{x^2}{4\alpha}}\Big)(\omega) = \mathrm{e}^{-\alpha\omega^2}.$
2. $\mathcal{F}(\mathrm{e}^{-\alpha|x|})(\omega) = \frac{\alpha}{\pi(\omega^2 + \alpha^2)}, \quad \mathcal{F}\left(\frac{2\alpha}{x^2 + \alpha^2}\right)(\omega) = \mathrm{e}^{-\alpha|\omega|}.$

6.1.2 Advanced properties

Remember that a **Dirac delta** is, as we have seen in Chapter 3, the limit of a sequence of functions with integral 1 that concentrate at the origin. We can compute its Fourier transform.

Theorem 6.3.

$$\mathcal{F}(\delta(x))(\omega) = \frac{1}{2\pi}\,, \qquad \mathcal{F}(\delta(x - x_0))(\omega) = \frac{\mathrm{e}^{i\omega x_0}}{2\pi}\,.$$

For two integrable functions defined on $\mathbb{R}$ we define the **convolution** as

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\,.$$

(This definition is different from that we used with the Laplace transform). Then we have:

Theorem 6.4.

$$\mathcal{F}(f * g)(\omega) = \widehat{f}(\omega)\widehat{g}(\omega).$$

There are different definitions for the Fourier transform, changing the coefficient and the exponent in the formula. This is done because we can simplify the formula of the transform or the formula for the inverse transform. It is common in mathematical texts to define the Fourier transform by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi i x \xi}\, dx\,.$$

6.2 Resolution of equations with the Fourier transform

6.2.1 Heat equation

We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}, & \text{if } x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & \text{if } x \in \mathbb{R}\,. \end{cases}$$

If we apply the Fourier transform in the variable x, we obtain

$$\begin{cases} \dfrac{\partial \widehat{u}}{\partial t}(\omega,t) = -k\omega^2 \widehat{u}(\omega,t), & t > 0, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

This is an ODE in t, that we solve separating variables,

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\, \mathrm{e}^{-k\omega^2 t}.$$

Now we antitranform. We use the Fourier inverse transform of the gaussian and the Fourier transform of the convolution,

$$\mathcal{F}^{-1}\left(\mathrm{e}^{-k\omega^2 t}\right) = \sqrt{\frac{\pi}{kt}}\, \mathrm{e}^{-\frac{x^2}{4kt}} \quad \Longrightarrow \quad u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4kt}}\, f(y)\, dy\,.$$

From this expression we deduce the Green's function

$$G(x,y,t) = \frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-\frac{(x-y)^2}{4kt}}.$$

Observe that $G(x,y,t) = \mathcal{K}(x-y,t)$, where

$$\mathcal{K}(x,t) = \frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-\frac{x^2}{4kt}}$$

is known as **Gauss' kernel.**

6.2.2 Laplace equation in the upper half-plane

The problem is described by;

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \\ \lim\limits_{y\to\infty} u(x,y) = 0, & x \in \mathbb{R}. \end{cases}$$

If we transform by Fourier in the x variable we obtain

$$\begin{cases} -\omega^2 \widehat{u}(\omega,y) + \dfrac{\partial \widehat{u}}{\partial y^2}(\omega,y) = 0, & \omega \in \mathbb{R},\ y > 0, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega), & \omega \in \mathbb{R}, \\ \lim\limits_{y\to\infty} \widehat{u}(\omega,y) = 0, & \omega \in \mathbb{R}. \end{cases}$$

Again, we have an ODE, now in the y variable. Its solution is $\widehat{u}(\omega,y) = A\mathrm{e}^{|\omega|y} + B\mathrm{e}^{-|\omega|y}$, and using the boundary conditions it is simply:

$$\widehat{u}(\omega,y) = \widehat{f}(\omega)\mathrm{e}^{-|\omega|y}.$$

Using now the inverse Fourier transform of the exponential and the Fourier transform of the convolution,

$$\mathcal{F}^{-1}\left(\mathrm{e}^{-|\omega|y}\right) = \frac{2y}{x^2+y^2} \quad \Longrightarrow \quad u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2} f(s)\, ds\,.$$

In this case the Green's function is

$$G(x,s,y) = \frac{1}{\pi}\frac{y}{(x-s)^2+y^2}.$$

As before,

$$G(x,s,y) = \mathcal{P}(x-s,y)$$

where

$$\mathcal{P}(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

is known as **Poison's kernel.**

6.3 Fourier transform in several variables

6.3.1 Definition and basic properties

For an integrable function f on $\mathbb{R}^n$, we define its **Fourier transform** by

$$\widehat{f}(\vec{\omega}) = \mathcal{F}(f)(\vec{\omega}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\vec{x})\, \mathrm{e}^{i\vec{\omega}\cdot\vec{x}}\, d\vec{x}\,,$$

where $\vec{\omega}\cdot\vec{x} = \omega_1 x_1 + \cdots + \omega_n x_n$ and $d\vec{x} = dx_1 \cdots dx_n$.

The properties of the Fourier transform in several variables are similar, with obvious modifications, to the properties we already know in dimension one.

Theorem 6.5.

1. $f(\vec{x}) = \mathcal{F}^{-1}(\mathcal{F}(f))(\vec{x}) = \displaystyle\int_{\mathbb{R}^n} \widehat{f}(\vec{\omega})\, \mathrm{e}^{-i\vec{\omega}\cdot\vec{x}}\, d\vec{\omega}$.
2. $\mathcal{F}(af + bg)(\vec{\omega}) = a\widehat{f}(\vec{\omega}) + b\widehat{g}(\vec{\omega})$.
3. $\mathcal{F}(f(\vec{x} - \vec{x}_0))(\vec{\omega}) = \mathrm{e}^{i\vec{\omega}\cdot\vec{x}_0}\widehat{f}(\vec{\omega})$.
4. $\mathcal{F}(f(\alpha\vec{x}))(\vec{\omega}) = \dfrac{1}{\alpha}\,\widehat{f}\left(\dfrac{\vec{\omega}}{\alpha}\right), \quad \alpha \in \mathbb{R}\setminus\{0\}$.
5. $\mathcal{F}\left(\dfrac{\partial f}{\partial x_j}\right)(\vec{\omega}) = -i\omega_j \widehat{f}(\vec{\omega}), \quad j = 1, 2 \cdots, n$.
6. $\mathcal{F}\left(\Delta f\right)(\vec{\omega}) = -|\vec{\omega}|^2 \widehat{f}(\vec{\omega})$.
7. *If* $f = f(\vec{x}, t)$, *then* $\mathcal{F}\left(\dfrac{\partial f}{\partial t}\right)(\vec{\omega}) = \dfrac{\partial \widehat{f}}{\partial t}(\vec{\omega})$.
8. $\mathcal{F}(f * g)(\vec{\omega}) = \widehat{f}(\vec{\omega})\widehat{g}(\vec{\omega})$.
9. $\mathcal{F}(\delta(\vec{x}))(\vec{\omega}) = \dfrac{1}{(2\pi)^n}\,, \qquad \mathcal{F}(\delta(\vec{x} - \vec{x}_0))(\vec{\omega}) = \dfrac{\mathrm{e}^{i\vec{\omega}\cdot\vec{x}_0}}{(2\pi)^n}$.
10. $\mathcal{F}\big(\mathrm{e}^{-\alpha|\vec{x}|^2}\big)(\vec{\omega}) = \dfrac{1}{(4\pi\alpha)^{n/2}}\, \mathrm{e}^{-\frac{|\vec{\omega}|^2}{4\alpha}}, \quad \mathcal{F}\left(\left(\dfrac{\pi}{\alpha}\right)^{n/2} \mathrm{e}^{-\frac{|\vec{x}|^2}{4\alpha}}\right)(\vec{\omega}) = \mathrm{e}^{-\alpha|\vec{\omega}|^2}$.

It is necessary to remember the concept of **Dirac delta** that, for example in dimension $n = 2$, can be defined as

$$\delta(\vec{x}) = \lim_{\varepsilon\to 0} f_\varepsilon(\vec{x}), \qquad \text{where} \quad f_\varepsilon(x) = \begin{cases} \dfrac{1}{\pi\varepsilon^2}, & |\vec{x}| \le \varepsilon, \\ 0, & |\vec{x}| > \varepsilon, \end{cases}$$

and also to extend the concept of **convolution**, that in dimension n is

$$\left(f * g\right)(\vec{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\vec{x} - \vec{y})\, g(\vec{y})\, d\vec{y}\,.$$

6.3.2 Heat equation on $\mathbb{R}^n$

As an example we solve the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\Delta u, & \text{if } \vec{x} \in \mathbb{R}^n,\ t > 0, \\ u(\vec{x}, 0) = f(\vec{x}), & \text{if } \vec{x} \in \mathbb{R}^n\,. \end{cases}$$

When we transform by Fourier in the $\vec{x}$ variable we obtain

$$\begin{cases} \dfrac{\partial \widehat{u}}{\partial t}(\vec{\omega}, t) = -k|\vec{\omega}|^2 \widehat{u}(\vec{\omega}, t), \\ \widehat{u}(\vec{\omega}, 0) = \widehat{f}(\vec{\omega}). \end{cases}$$

The solution of the ODE in t is

$$\widehat{u}(\vec{\omega}, t) = \widehat{f}(\vec{\omega})\, \mathrm{e}^{-k|\vec{\omega}|^2 t},$$

and antitransforming we arrive to the solution in terms of the Gauss' kernel,

$$u(\vec{x}, t) = \int_{\mathbb{R}^n} G(\vec{x}, \vec{y}, t)\, f(\vec{y})\ d\vec{y} = \frac{1}{(4k\pi t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|\vec{x}-\vec{y}|^2}{4kt}}\ f(\vec{y})\ d\vec{y}\,.$$

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