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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Problems

Degree in Biomedical Engineering

Chapter 1

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1 First order differential equations

1.1 Elementary resolution methods

Problem 1.1.1 Express the following physical principles using differential equations.

a) The rate at which a radioactive substance decays is proportional to the quantity of the substance N.

b) The population P of a city grows at a rate proportional to the population and to the difference between 200.000 and the population.

c) The rate of change in vapor pressure P of a certain substance with respect to the temperature T is proportional to the pressure and inversely proportional to the square of the temperature.

d) Force is equal to mass multiplied by acceleration.

Problem 1.1.2 Find the differential equation that satisfies the indicated family of curves.

a) The parabolas with equation $y = cx + x^2$.

b) The straight lines with equation $y = cx + c^2$.

c) The circumferences on the plane with center on a point on the x axis.

d) The curves with equation $y = \dfrac{1 + \mathrm{c}e^{2x}}{1 - \mathrm{c}e^{2x}}$.

e) The cardioids $r = a(1 - \cos\theta)$.

Problem 1.1.3 Find the general solutions to the following differential equations using separation of variables.

i) $dy = \mathrm{e}^{3x-2y}\,dx\,,$ *ii*) $xyy' = (x+1)(y+1)\,,$

iii) $\mathrm{e}^{x}yy' = \mathrm{e}^{-y} + \mathrm{e}^{-2x-y}\,,$ *iv*) $(4y + yx^2)\,dy - (2x + xy^2)\,dx = 0\,,$

v) $(1 + x^2 + y^2 + x^2y^2)\,dy = y^2\,dx\,,$ *vi*) $2y' - \dfrac{1}{y} = \dfrac{2x}{y}\,,$

vii) $y' = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}\,,$ *viii*) $y' = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3},$

ix) $y'\sec y + \sin(x - y) = \sin(x + y)\,,$ *x*) $y' + xy = y\,.$

Problem 1.1.4 Find the general solutions of the following homogeneous differential equations.

i) $(x^2 - 2y^2)\,dx + xy\,dy = 0\,,$ *ii*) $x^2y' - 3xy - 2y^2 = 0\,,$

iii) $x^2y' = xy + 3(x^2 + y^2)\arctan\dfrac{y}{x}\,,$ *iv*) $xy'\sin\dfrac{y}{x} = x + y\sin\dfrac{y}{x}\,,$

v) $xy' = y + 2x\mathrm{e}^{-y/x}\,,$ *vi*) $-y\,dx + (x + \sqrt{xy})\,dy = 0\,,$

vii) $(5y^4 + 2x^3y)\,dx = x(x^3 + 4y^3)\,dy\,,$ *viii*) $y' = \dfrac{y}{x} + \dfrac{x}{y}\,,$

ix) $y\dfrac{dx}{dy} = x + 4y\mathrm{e}^{-2x/y}\,,$ *x*) $\left(y + x\,\mathrm{cotg}\,\dfrac{y}{x}\right)dx = x\,dy\,.$

Problem 1.1.5

a) Given the equation $y' = f\Big(\dfrac{Ax+By+C}{Dx+Ey+F}\Big)$, if $AE \neq BD$, show that there is a shift $w = x - x_0$, $z = y - y_0$, that reduces that equation to a homogeneous one.

b) If $AE = BD$ in the above equation, determine which change of variables reduces the equation to a separable one.

Problem 1.1.6 Apply the previous problem to solve the following equations.

$$i)\quad y' = (2x+2y+4)^2\,, \qquad ii)\quad y' = \cos(x-y+5)\,,$$

$$iii)\quad y' = \frac{x+y+4}{x-y-6}\,, \qquad iv)\quad y' = \frac{x+y+4}{x+y-6}\,.$$

Problem 1.1.7 Determine if the following differential equations are exact. In case they are, find their general solution.

$$i)\quad (\sin x \tan y + 1)\,dx + \cos x \sec^2 y\,dy = 0\,, \qquad ii)\quad (y - x^3)\,dx + (x + y^3)\,dy = 0\,,$$

$$iii)\quad (2y^2 - 4x - 5)\,dx = (4 - 2y + 4xy)\,dy\,, \qquad iv)\quad \frac{4y^2 - 2x^2}{4xy^2 - x^3}\,dx + \frac{8y^2 - x^2}{4y^3 - x^2 y}\,dy = 0\,,$$

$$v)\quad (\sin x \sin y - x\mathrm{e}^y)\,dy = (\mathrm{e}^y + \cos x \cos y)\,dx\,, \qquad vi)\quad (1+y)\,dx + (1-x)\,dy = 0\,,$$

$$vii)\quad (2xy^3 + y\cos x)\,dx + (3x^2y^2 + \sin x)\,dy = 0\,, \qquad viii)\quad dx = \frac{y}{1 - x^2y^2}\,dx + \frac{x}{1 - x^2y^2}\,dy\,,$$

$$ix)\quad (xy^2 + 3x^2y)\,dx + (x^3 + x^2y)\,dy = 0\,.$$

Problem 1.1.8 Let $f(y)$ be a differentiable function and different from zero, and consider the equation

$$f(y)\,dx - (x+y)\,dy = 0.$$

a) Determine the function $f(y)$ for which the equation is exact.

b) Calculate the integrating factor of the form $\mu(y)$ in terms of $f(y)$.

c) Solve the equation if $f(y) = y$.

Problem 1.1.9 Solve the following equations using an integrating factor of only one variable.

$$i)\quad (4x + 3y^3)\,dx + 3xy^2\,dy = 0, \qquad ii)\quad (4xy^2 + y)\,dx + (6y^3 - x)\,dy = 0,$$

$$iii)\quad 2x\,dx + x^2 \operatorname{cotg} y\,dy = 0, \qquad iv)\quad (y^2 + 2xy)\,dx - x^2\,dy = 0,$$

$$v)\quad y^2 \cos x\,dx + (4 + 5y\sin x)\,dy = 0, \qquad vi)\quad 6x^3y\,dx + x^4\,dy = 0.$$

Problem 1.1.10 Solve the equation $(7x^4y - 3y^8) + (2x^5 - 9xy^7)y' = 0$ using an integrating factor of the form $x^n y^m$.

Problem 1.1.11 Solve the following linear differential equations:

i) $xy' - 3y = x^4$, *ii)* $y' + y = \dfrac{1}{1 + \mathrm{e}^{2x}}$,

iii) $(1 + x^2)\, dy + 2xy\, dx = \operatorname{cotg} x\, dx$, *iv)* $y' + y = 2x\mathrm{e}^{-x} + x^2$,

v) $y' + y \operatorname{cotg} x = 2x \operatorname{cosec} x$, *vi)* $(2y - x^3)\, dx = x\, dy$,

vii) $y - x + xy \operatorname{cotg} x + xy' = 0$, *viii)* $\dfrac{dx}{dy} - 2xy = 6y\mathrm{e}^{y^2}$.

Hint: *viii)* This equation is linear if x is considered as a dependent variable, $x = x(y)$.

Problem 1.1.12 Solve the following Bernoulli type equations.

i) $xy' + y = xy^2$, *ii)* $xy' + y = x^4y^3$,

iii) $xy^2y' + y^3 = x\cos x$, *iv)* $xy' = 6y + 12x^4y^{2/3}$,

v) $y' - 5y = \dfrac{5}{2}xy^3$.

Problem 1.1.13 Use two different methods to solve the equation $2x^2y - x^3y' = y^3$.

Problem 1.1.14 Show that equations of the form $y' + P(x)y = Q(x)y \log y$ are solved using the substitution $z = \log y$. Apply this method to solve the equation

$$xy' = 2x^2y + y \log y\,.$$

Problem 1.1.15 Solve the equation

$$(3x + 2y + y^2)dx + (x + 4xy + 5y^2)dy = 0$$

using an integrating factor of the form $\mu(x, y) = \phi(x + y^2)$.

Problem 1.1.16 Classify and solve the following equations

i) $y' = \dfrac{y}{x} + \dfrac{x^2}{y^2}$; *ii)* $xy' + y = \sin 2x$;

iii) $(2y + yx^2)\, dy + (3x + xy^2)\, dx = 0$; *iv)* $y\mathrm{e}^{xy}\, dx + (x\mathrm{e}^{xy} + \cos y)\, dy = 0$;

v) $3xy^2y' + y^3 = x \sin x$; *vi)* $x^4yy' + \left(\dfrac{3}{2}y^2 + 1\right)x^3 = 1$.

Problem 1.1.17 Solve the following equations by order reduction:

i) $x^2y'' + xy' = 0$; *ii)* $yy'' - (y')^2 = 0$;

iii) $yy'' + (y')^2 = 0$; *iv)* $xy'' = y' + (y')^3$;

v) $y'' - k^2y = 0$; *vi)* $xy'' + y' = 4x$.

Problem 1.1.18
Solve the following Ricatti equation,

$$y' = 1 + x^2 + y^2 - 2xy,$$

knowing that a particular solution is $y_1(x) = x$.

Problem 1.1.19 Given the following Ricatti equation,

$$y' = \frac{1}{3}x^2y^2 + \left(\frac{2}{x} - 2x^4\right)y + 3x^6,$$

solve it looking for a particular solution of the form $y_1(x) = kx^\alpha$.

1.2 Applications

Problem 1.2.1 Verify that the following families of curves are orthogonal.

$$i)\quad \mathcal{C}_1 = \{y = c_1x\} \text{ and } \mathcal{C}_2 = \{x^2 + y^2 = c_2\},$$

$$ii)\quad \mathcal{C}_1 = \{y = c_1x^3\} \text{ and } \mathcal{C}_2 = \{x^2 + 3y^2 = c_2\},$$

$$iii)\quad \mathcal{C}_1 = \{x^2 + y^2 = 2c_1x\} \text{ and } \mathcal{C}_2 = \{x^2 + y^2 = 2c_2y\}.$$

Problem 1.2.2 Find the families of curves orthogonal to each one of the following families.

i) $y = cx^2$	*ii)* $cx^2 + y^2 = 1$	*iii)* $y = ce^{-x}$
iv) $y^2 = cx^3$	*v)* $\sinh y = cx$	*vi)* $y = x/(1 + cx)$
vii) $y = 1/\log(cx)$	*viii)* $x^{1/3} + y^{1/3} = c$	*ix)* $xy = c$

Problem 1.2.3 Find the curves such that the distances along the tangent line from the point of tangency to the intercept with each axis are equal.

Problem 1.2.4 Determine a curve contained in the first quadrant with the property that given any point (x, y) of the curve, it divides every rectangle with vertices $(0, 0)$, $(0, y)$, $(x, 0)$, (x, y), where the areas of the two parts are in a 1:2 proportion.

Problem 1.2.5 The logistic model of population growth is governed by the differential equation $P' = aP - bP^2$.

a) Determine the population at an instant t given that the initial population is M.

b) Calculate the fish population in a pond in the year 1998 given that it started with 1.000 fish in the year 1990 and it was estimated that during the years 1992 and 1994 it was of 3.000 and 5.000 fish, respectively.

c) What is the long term equilibrium population?

Problem 1.2.6 A snowball melts down in such a way that the rate of change of its volume is proportional to its surface area. The snowball's diameter is initially 4cm of length and 30 minutes later its diameter reduces to 3cm.

a) When will the diameter be 2cm of length?

b) When will the snowball melt down entirely?

Problem 1.2.7 Radiocarbon ^{14}C on alive wood decays at a rate of 15,30 disintegrations per minute (dpm) per gram of carbon. If the half-life of ^{14}C is 5.600 years, estimate the age of each of the following objects discovered by archeologists and analysed by radioactive decay in 1950.

a) A fragment of a chair's leg inside Tutankhamon's tomb, found in the Valley of the Kings (Egypt): d =10,14 dpm.

b) An arrow found in Leonard Rock Shelter (Nevada, USA): d =6,42 dpm.

c) Manure of a giant sloth found inside Gypsum Cave (Nevada, USA): d =4,17 dpm.

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