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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Problems

Degree in Biomedical Engineering

Chapter 2

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2 Linear equations of higher order

2.1 Linear equations of order n with constant coefficients

Problem 2.1.1 Solve the following equations of order two

$$i)\quad y'' + 2y' - 3y = 0\,, \qquad ii)\quad y'' = 4y\,,$$

$$iii)\quad y'' + 4y' - 5y = 0\,, \qquad iv)\quad y'' + 8y = 0\,,$$

$$v)\quad y'' + 4y' + 5y = 0\,, \qquad vi)\quad 2y'' + 2y' + 3y = 0\,,$$

Problem 2.1.2 Find the general solution of the following nonhomogeneous equations:

$$i)\quad y'' + 3y' - 10y = 6\mathrm{e}^{4x}, \qquad ii)\quad y'' - 2y' + 5y = 25x^2 + 12,$$

$$iii)\quad y'' - y' - 6y = 20\mathrm{e}^{-2x}, \qquad iv)\quad y'' - 3y' + 2y = 10\sin 2x,$$

$$v)\quad y'' - 2y' + y = \mathrm{e}^{x} + \mathrm{e}^{-2x}, \qquad vi)\quad y'' - y = x\sin x,$$

$$vii)\quad y'' - 6y' + 9y = 5\mathrm{e}^{x}\sin x, \qquad viii)\quad y'' + y' + y + 1 = \sin x + x + x^2.$$

Problem 2.1.3 Find the general solution of the following equations:

$$i)\quad y''' - 3y'' + 2y' = 0, \qquad ii)\quad y''' - y = 0,$$

$$iii)\quad y^{(4)} - 2a^2y'' + a^4y = 0, \qquad iv)\quad y''' - 3y' + 2y = 3x^2 + x,$$

$$v)\quad y^{(4)} - 2y''' + 2y'' - 2y' + y = e^{x}.$$

Problem 2.1.4 Determine the linear differential homogeneous equation from the fundamental system of solutions that are given:

$$i)\quad \{1, \mathrm{e}^{x}\}, \qquad ii)\quad \{\sin 3x,\, \cos 3x\},$$

$$iii)\quad \{\mathrm{e}^{-2x},\, x\mathrm{e}^{-2x}\}, \qquad iv)\quad \{\mathrm{e}^{2x},\, \sin x,\, \cos x\},$$

$$v)\quad \{\mathrm{e}^{x},\, x\mathrm{e}^{x},\, x^2\mathrm{e}^{x}\}.$$

Problem 2.1.5 For the equation $y'' + ay' + by = f$, we know that the following functions are solutions: $y_1(x) = \mathrm{e}^{-x} + \mathrm{e}^{-3x}$, $y_2(x) = \mathrm{e}^{-x} - \mathrm{e}^{-3x}$, $y_3(x) = \mathrm{e}^{-x} + x\mathrm{e}^{-3x}$. Determine a, b and f and obtain the general solution.

Problem 2.1.6 Consider the differential equation

$$x'''(t) + 2x''(t) - x'(t) - 2x(t) = 40\mathrm{e}^{3t}.$$

a) Check that $x(t) = \mathrm{e}^{3t}$ is a solution.

b) Find the solution that satisfies $x(0) = 2,\; x'(0) = x''(0) = 7.$

Problem 2.1.7 Find the solutions of the following initial value problems

i) $x'' + 4x' + 3x = 0,\qquad x(0) = 0,\ x'(0) = 1;$

ii) $x''' + 6x'' + 12x' + 8x = 0,\qquad x(1) = -2,\ x'(1) = x''(1) = 0;$

iii) $x^{(5)} + 4x^{(4)} + 5x^{(3)} = 0,\qquad x(0) = x'(0) = x''(0) = 1,\ x^{(3)}(0) = x^{(4)}(0) = 0;$

iv) $x^{(5)} + 4x^{(4)} + 3x^{(3)} = 0,\qquad x(0) = x'(0) = x''(0) = 0,\ x^{(3)}(0) = 1,\ x^{(4)}(0) = -1.$

Problem 2.1.8 Solve the following problem:

$$\begin{cases} y''' - 2y'' + y' = 2 - 24\mathrm{e}^{x} + 40\mathrm{e}^{5x}, \\ y(0) = \dfrac{1}{2},\ y'(0) = \dfrac{5}{2},\ y''(0) = -\dfrac{9}{2}. \end{cases}$$

Problem 2.1.9 Find a particular solution to each one of the following equations:

i) $y'' + 4y = \operatorname{tg} 2x,\qquad$ *ii)* $y'' + 2y' + y = \mathrm{e}^{-x}\log x,$

iii) $y'' - 2y' - 3y = 64x\mathrm{e}^{-x},\qquad$ *iv)* $y'' + 2y' + 5y = \mathrm{e}^{-x}\sec 2x.$

Problem 2.1.10 Solve the following Euler differential equations:

i) $x^2y'' - 4xy' + 4y = 0\,,\qquad$ *ii)* $x^2y'' + xy' - n^2y = 0\,,\quad n \neq 0\,,$

iii) $x^2y'' + xy' = 0\,,\qquad$ *iv)* $x^3y''' + 3x^2y'' = 0\,,$

v) $x^3y''' + x^2y'' - 2xy' + 2y = 0\,,\qquad$ *vi)* $x^3y''' + 2x^2y'' + xy' - y = 0\,,$

vii) $x^2y'' + 3xy' + y = 3x^2\,,\qquad$ *viii)* $x^3y''' + xy' - y = 3x^4\,.$

Problem 2.1.11 Find a second independent solution of the Bessel equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0,$$

knowing that a solution is $y_1(x) = \dfrac{\cos x}{\sqrt{x}}$.

Problem 2.1.12 Find the general solution of the following equations using a given solution of the homogeneous equation:

i) $(x^2 - 1)y'' - 2xy' + 2y = (x^2 - 1)^2,\qquad y_1(x) = x\,,$

ii) $(x^2 + x)y'' + (2 - x^2)y' - (2 + x)y = x(x + 1)^2,\qquad y_1(x) = 1/x\,.$

2.2 Applications

Problem 2.2.1 To determine the resistance of a small sphere moving with constant speed in a viscous fluid, it is necessary to solve the Euler differential equation

$$t^3x'''' + 8t^2x''' + 8tx'' - 8x' = 0$$

Find its general solution.

Problem 2.2.2 The differential equation that models an RLC circuit is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + C^{-1}I = \frac{dE}{dt},$$

where $E = 10\sin 2t$ stands for the electromotive force of the circuit and $R = 20\Omega$, $L = 10h$, $C = 0.01f$. Determine the magnitude of the current in the circuit if the initial current and the initial charge of the condenser are null,

Problem 2.2.3 Consider a spring such that when an external force of $100Nw$ is applied it stretches $2m$. When it is placed on a table and has attached a mass of $0.5kg$ it stretches $50cm$ and its velocity (in the opposite direction of the stretching) is $15m/s$. If the friction with the table is $6kg/s$, compute the position of the mass along the time.

Problem 2.2.4 Consider the spring-mass problem to which an external periodic force is applied

$$\begin{cases} 2u'' = -8u + \cos\omega t, \\ u(0) = u'(0) = 0. \end{cases}$$

a) Solve the problem for $|\omega| \neq 2$.

b) Calculate the limit of the solution when $\omega \to 2$.

c) Solve the problem for $\omega = 2$.

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