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Departamento de Matemáticas

# DIFFERENTIAL EQUATIONS. Problems

**Degree in Biomedical Engineering**

**Chapter 3**

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# 3 Laplace transform

## 3.1 Properties of the Laplace transform

**Problem 3.1.1** Prove the following properties of the *gamma* function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0.$$

*a*) $\Gamma(1) = \Gamma(2) = 1$; $\Gamma(1/2) = \sqrt{\pi}$.

*b*) $\Gamma(x+1) = x\Gamma(x)$.

*c*) Deduce from the above that $\lim\limits_{x\to 0^+} \Gamma(x) = +\infty$.

*d*) If $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

*e*) If $n \in \mathbb{N}$, $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}$.

Hint: for *a*) use that $\int_0^\infty e^{-z^2}\, dz = \frac{1}{2}\sqrt{\pi}$.

**Problem 3.1.2**

*a*) Write the Laplace transform of the function $f(x) = x^\alpha$, $(\alpha > -1)$, with the help of the *Gamma* function.

*b*) Find the Laplace transform of the following functions:

$$i)\ f(x) = \sqrt{x} e^{3x}, \qquad ii)\ f(x) = \frac{e^x}{\sqrt{x}}.$$

**Problem 3.1.3** Prove the following properties of the Laplace transform:

*a*) $L(e^{-at} f(t))(s) = L(f)(s+a), \qquad a \in \mathbb{R}.$

*b*) $L(f(at))(s) = \frac{1}{a}\, L(f(t))\left(\frac{s}{a}\right), \qquad a > 0.$

**Problem 3.1.4** Using the above properties, calculate the Laplace transform of the following functions, and indicate in each case their domain.

*i*) $f(x) = 2 + x - 4x^2 + 3x^3$,

*ii*) $f(x) = x\, e^{ax}, \quad (a \in \mathbb{R})$,

*iii*) $f(x) = e^x \sqrt{x^3}$,

*iv*) $f(x) = x^b e^{ax}, \quad (a \in \mathbb{R},\ b > -1)$,

*v*) $f(x) = 3e^{2x} + x^2 e^{-x} - 2x^3 e^x$,

*vi*) $f(x) = \sin(ax), \quad (a \in \mathbb{R})$,

*vii*) $f(x) = \cos(ax), \quad (a \in \mathbb{R})$,

*viii*) $f(x) = e^{-ax}\cos(bx), \quad (a, b \in \mathbb{R})$,

*ix*) $f(x) = e^{-ax}\sin(bx), \quad (a, b \in \mathbb{R})$,

*x*) $f(x) = \sin^2 x$,

*xi*) $f(x) = \cos^2 x$,

*xii*) $f(x) = \sin^3 x$,

*xiii*) $f(x) = \cos^3 x$.

Hint: *xii*) $4\sin^3 x = 3\sin x - \sin 3x$; *xiii*) $4\cos^3 x = 3\cos x + \cos 3x$.

**Problem 3.1.5** Let $f$ be a continuous function on $[0,\infty)$ with at most exponential growth.

$a)$ Suppose that the derivative can be taken under the integral sign. Prove that $L(f)$ is differentiable and that

$$\frac{d}{ds}[L(f)(s)] = -L(t\, f(t))(s)\,.$$

$b)$ Furthermore, assuming that the derivative can be taken under the integral sign, prove that$L(f)$ has derivatives of all orders and that

$$\frac{d^n}{ds^n}[L(f)(s)] = (-1)^n L(t^n f(t))(s)\,.$$

**Problem 3.1.6** Obtain the Laplace transform for the following functions, indicating in each case the domain:

$$i)\ f(x) = x\cos(ax), \qquad ii)\ f(x) = x^2\sin(ax).$$

**Problem 3.1.7** Let us define the Heaviside (or step) function as

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

$a)$ Prove that

$$L(f(t-a)\,H(t-a)) = \mathrm{e}^{-as} L(f(t))(s), \qquad a \geq 0.$$

$b)$ Calculate $L(g)(s)$ for

$$g(t) = \begin{cases} 0, & 0 < t < 3, \\ t^2, & t > 3. \end{cases}$$

Hint: In $b)$ write $t^2$ in powers of $(t-3)$.

**Problem 3.1.8** Find the function whose Laplace transform is

$$i)\ \frac{1}{s^2-1}, \qquad ii)\ \frac{1}{(s+1)^2},$$

$$iii)\ \frac{1}{s(s+1)^2}, \qquad iv)\ \frac{1}{s^n}, \quad (n \in \mathbb{N}),$$

$$v)\ \frac{1}{(s-1)^2(s^2+1)}, \qquad vi)\ \frac{4s+12}{s^2+8s+16},$$

$$vii)\ \frac{s\,\mathrm{e}^{-\pi s/2}}{s^2+a^2}, \qquad viii)\ \frac{1}{\sqrt{s}},$$

$$ix)\ \frac{1}{s^4-s^2}, \qquad x)\ \frac{s+3}{(s+1)^2+9} + \frac{\mathrm{e}^{-2s}}{s^2+5s+6}\,.$$

**Problem 3.1.9** Prove that if $f$ is continuous in $[0,\infty)$ and has at most exponential growth, then the same is valid for the function $g(x) = \int_0^x f(t)\,dt$, and that

$$L(g)(s) = \frac{1}{s}\,L(f)(s)\,.$$

**Problem 3.1.10** Calculate the Laplace transform of the following function

$$f(x) = x \int_0^x \mathrm{e}^{-at} \sin(bt)\, dt\,, \qquad a, b \in \mathbb{R}\,.$$

**Problem 3.1.11** Let $f : (0, \infty) \longrightarrow \mathbb{R}$ be a piecewise continuous function and with at most exponential growth.

*a*) Prove that if $f$ is periodic with period $P$, that is, $f(x+P) = f(x)$ for every $x > 0$, then

$$L(f)(s) = \frac{1}{1 - \mathrm{e}^{-Ps}} \int_0^P \mathrm{e}^{-st} f(t)\, dt\,.$$

*b*) As an application of the formula in the previous part, calculate the Laplace transform of

*i*) $f(x) = x - [x]$, where $[x]$ denotes the integer part of $x$;

*ii*) $f(x) = (-1)^{[x]}$.

Hint: in *a*) divide the integral that defines $L(f)$ into two parts, one from 0 to $P$, and the other from $P$ to infinity, and make an adequate change of variable in the second integral such that you can use the fact that $f$ is periodic.

## 3.2 Linear equations and systems.

**Problem 3.2.1** Solve the following initial value problems

*i*) $\begin{cases} y' - 3y = \mathrm{e}^{2t}, \\ y(0) = 1; \end{cases}$ *ii*) $\begin{cases} y' + 3y = \sin 2t, \\ y(0) = 0; \end{cases}$

*iii*) $\begin{cases} y' - 5y = \cos 3t, \\ y(0) = 1/2; \end{cases}$ *iv*) $\begin{cases} y' - 5y = \mathrm{e}^{5t}, \\ y(0) = 1/2; \end{cases}$

*v*) $\begin{cases} y'' + 2y' + y = \mathrm{e}^{-t}, \\ y(0) = 0,\ y'(0) = 1; \end{cases}$ *vi*) $\begin{cases} y'' - y = \mathrm{e}^{2t}, \\ y(0) = 0,\ y'(0) = 1; \end{cases}$

*vii*) $\begin{cases} y'' + 16y = \cos t, \\ y(0) = 0,\ y'(0) = 1; \end{cases}$ *viii*) $\begin{cases} y'' + 2y' + y = \mathrm{e}^{-3t}, \\ y(0) = 1,\ y'(0) = 0; \end{cases}$

*ix*) $\begin{cases} y'' - 6y' + 9y = t^2\mathrm{e}^{3t}, \\ y(0) = 2,\ y'(0) = 6; \end{cases}$ *x*) $\begin{cases} y'' + 4y' + 6y = 1 + \mathrm{e}^{-t}, \\ y(0) = 0,\ y'(0) = 0; \end{cases}$

*xi*) $\begin{cases} y'' - y' = \mathrm{e}^{t} \cos t, \\ y(0) = y'(0) = 0; \end{cases}$ *xii*) $\begin{cases} y'' - 2y' + 3y = \mathrm{e}^{t} \sin t, \\ y(0) = 0,\ y'(0) = 1. \end{cases}$

**Problem 3.2.2** Solve each one of the following problems using two different methods:

*i*) $\begin{cases} y'' + 4y' + 8y = \mathrm{e}^{-t}, \\ y(0) = y'(0) = 0; \end{cases}$ *ii*) $\begin{cases} y'' - 2y' + y = 1 + x\mathrm{e}^{2x}, \\ y(0) = y'(0) = 0\,. \end{cases}$

**Problem 3.2.3** Solve the following initial value problems for $t > 0$:

$$i)\begin{cases} y'''(t) - 4y''(t) - 5y'(t) = 3, \\ y(0) = y''(0) = 0,\ y'(0) = 1; \end{cases}$$

$$ii)\begin{cases} x'''(t) + x''(t) - 6x'(t) = 0, \\ x(0) = x'(0) = 0,\ x''(0) = 1. \end{cases}$$

**Problem 3.2.4** Solve the following problem:

$$\begin{cases} y'' + 16y = \cos 4t, \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Hint: Verify that $L^{-1}(\frac{s}{(s^2+a^2)^2}) = \frac{1}{2a} t \sin(at)$.

**Problem 3.2.5** Solve for $\omega \neq \omega_0$ the initial value problem

$$\begin{cases} x'' + \omega_0^2\, x = k \sin \omega t, \quad t > 0, \\ x(0) = x'(0) = 0, \end{cases}$$

which describes the forced oscillations of a mass on an undamped spring. What happens if $\omega = \omega_0$?
Hint: Verify that $L^{-1}(\frac{1}{(s^2+a^2)^2}) = \frac{1}{2a^3}(\sin at - at \cos at)$.

**Problem 3.2.6** Find the function $f(x)$ that satisfies the initial condition $f(0) = -1$ and the equation

$$f'(x) + \int_0^x f(t)\, dt = x^2 - x + 2.$$

**Problem 3.2.7** Solve the following initial value problems:

$$i)\begin{cases} x' = 4x - y, \\ y' = 2x + y, \\ x(0) = 0,\ y(0) = 1; \end{cases} \quad ii)\begin{cases} x' = 5x + 4y, \\ y' = -x + y, \\ x(0) = 0,\ y(0) = 1. \end{cases} \quad iii)\begin{cases} x' = -x - y, \\ y' = 2x - y, \\ x(0) = 1,\ y(0) = 2; \end{cases}$$

$$iv)\begin{cases} x' = x - 2y + 2, \\ y' = 5x - y + 1, \\ x(0) = y(0) = 0; \end{cases} \quad v)\begin{cases} x' = x - 2y - t, \\ y' = 2x - 3y - t, \\ x(0) = y(0) = 1; \end{cases} \quad vi)\begin{cases} x' = 2x + y, \\ y' = x + 2y, \\ x(0) = 0,\ y(0) = 1; \end{cases}$$

$$vii)\begin{cases} x' = x + 2y, \\ y' = 4x + 3y, \\ x(0) = 1,\ y(0) = 0; \end{cases} \quad viii)\begin{cases} x' = -3x + 2y, \\ y' = -2x - 3y, \\ x(0) = 2,\ y(0) = -1; \end{cases} \quad ix)\begin{cases} x' = -x + 2y + \mathrm{e}^{2t}, \\ y' = 4x + y - 2\mathrm{e}^{2t}, \\ x(0) = y(0) = 0. \end{cases}$$

**Problem 3.2.8** Solve the following first order linear systems of differential equations turning each one of them into a second order linear equation.

$$i)\begin{cases} x' = 3x - 4y, \\ y' = x - y, \\ x(0) = 0,\ y(0) = 2; \end{cases} \quad ii)\begin{cases} x' - 6x + 3y = 8\mathrm{e}^t, \\ y' - 2x - y = 4\mathrm{e}^t, \\ x(0) = -1,\ y(0) = 0. \end{cases}$$

**Problem 3.2.9** Solve the following integral equation using the Laplace transform of the convolution:

$$f(t) = 4t + \int_0^t f(t-r)\sin r\,dr\,.$$

**Problem 3.2.10** Solve the integro-differential equation:

$$f'(x) = 1 - \int_0^x f(x-r)\mathrm{e}^{-2r}dr, \quad f(0) = 1.$$

**Problem 3.2.11** Find the solution of the following integro-differential equation:

$$x - \frac{1}{4}f(x) = \int_0^x (x-u)f(u)\mathrm{d}u.$$

**Problem 3.2.12** Solve the following problems:

$$i)\ \left\{\begin{array}{l} y'' + y' = \left\{\begin{array}{ll} t+1, & 0<t<1, \\ 3-t, & t>1, \end{array}\right. \\ y(0) = -1,\, y'(0) = 0; \end{array}\right. \qquad ii)\ \left\{\begin{array}{l} y'' + 4y = \left\{\begin{array}{ll} \cos 2t, & 0<t<2\pi, \\ 0, & t>2\pi, \end{array}\right. \\ y(0) = y'(0) = 0. \end{array}\right.$$

**Problem 3.2.13** Obtain the solution of the problem $\left\{\begin{array}{l} x'' - x = f(t), \\ x(0) = x'(0) = 0, \end{array}\right.$ where $f(t) = \mathrm{e}^t$ if $\pi/2 < t < \pi$, $f(t) = 0$ otherwise.

**Problem 3.2.14** Solve the following problems:

$$i)\ \left\{\begin{array}{l} y'' = \delta(t-a), \\ y(0) = 0,\, y'(0) = 0; \end{array}\right. \qquad ii)\ \left\{\begin{array}{l} y' + 8y = \delta(t-1) + \delta(t-2), \\ y(0) = 0; \end{array}\right.$$

$$iii)\ \left\{\begin{array}{l} y'' + y = 4\delta\left(t - \frac{3}{2}\pi\right), \\ y(0) = 0,\, y'(0) = 1; \end{array}\right. \qquad iv)\ \left\{\begin{array}{l} y'' + y = 1 + \delta(t-2\pi), \\ y(0) = 1,\, y'(0) = 0. \end{array}\right.$$

**Problem 3.2.15** The current on an electric circuit with inductance L and resistance R is given by

$$L\frac{dI}{dt} + RI = E\,.$$

where E is the electromotive force that is applied. If $I(0) = 0$, find I in the following cases: ($E_0$ is constant)

$$i)\ E(t) = E_0\delta(t), \qquad ii)\ E(t) = E_0\sin(\omega t).$$

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