uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Problems

Degree in Biomedical Engineering

Chapter 4

Elena Romera

Open Course Ware, UC3M
http://ocw.uc3m.es/matematicas

4 Method of separation of variables

4.1 Separation of variables

Problem 4.1.1 For each of the following PDE's, determine the ODE that are obtained when the method of separation of variables is applied:

$$i)\quad \frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad ii)\quad r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0,$$

$$iii)\quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}, \qquad iv)\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$v)\quad \frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \qquad vi)\quad \frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4},$$

$$vii)\quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}.$$

(k, v_0 and c are constants).

Problem 4.1.2 Use the method of separation of variables to solve the problem over the square $\{0 < x < \pi,\ 0 < y < \pi\}$:

$$\begin{cases} \Delta u = 0, \\ u(0,y) = u(\pi,y) = 0, \\ u(x,\pi) = 0, \\ u(x,0) = 2\sin 3x. \end{cases}$$

Problem 4.1.3 Solve the following problem over the square $\{0 < x < L,\ 0 < y < L\}$:

$$\begin{cases} \Delta\varphi = 0, \\ \dfrac{\partial\varphi}{\partial x}(0,y) = \dfrac{\partial\varphi}{\partial x}(L,y) = 0, \\ \varphi(x,0) = 0, \\ \varphi(x,L) = f(x). \end{cases}$$

Problem 4.1.4 Consider the Laplace equation over the rectangle $\{0 < x < L, 0 < y < H\}$ with boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial x}(L,y) = 0, \quad \frac{\partial u}{\partial y}(x,H) = f(x).$$

a) Obtain the compatibility condition.

b) Solve the problem using the method of separation of variables. Show that this method produces a solution only under the condition derived in *a*).

c) The solution of part *b*) contains an arbitrary constant. Determine it by considering the relation of the proposed problem with the condition for the heat equation over the rectangle with initial condition $u(x,y,0) = g(x,y)$.

Problem 4.1.5 Consider the problem corresponding to the temperature on a one-dimensional cylinder with absorbtion and external temperature 0 degrees Celsius:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2} - \alpha u, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L. \end{cases}$$

a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

b) Solve the time dependent problem. Analyze the temperature for large values of t ($t \to \infty$) and compare the solution with the solution of the previous item.

Problem 4.1.6 The previous problem is more delicate in the case of $\alpha < 0$. Solve the particular cases $\alpha = -\dfrac{3\pi}{2L^2}$ and $\alpha = -\dfrac{2\pi}{L^2}$.

Problem 4.1.7 Consider the problem corresponding to the vibration of a string with fixed ends

$$\begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} = T_0\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L, \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & 0 < x < L, \end{cases}$$

where $\rho > 0$ is the constant density of the string and $T_0 > 0$ is the tension.

a) Solve the problem by separation of variables and verify that the solution can be written as a sum of *harmonics*

$$u(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} \alpha_n \cos[\omega_n(t-\delta_n)]\sin(\sqrt{\lambda_n}\,x)$$

identifying λ_n, ω_n, α_n and δ_n.

b) Find the points of maximum amplitude for each one of the harmonics, called *antinodes.*

c) Find the *energy* for each harmonic

$$E_n = \frac{\rho}{2}\int_0^L \left(\frac{\partial U_n}{\partial t}\right)^2 dx + \frac{T_0}{2}\int_0^L \left(\frac{\partial U_n}{\partial x}\right)^2 dx$$

and verify that it is constant in time.

d) A string pulled at the middle some length $A > 0$ from the horizontal and then released corresponds to the initial conditions

$$f(x) = \begin{cases} \dfrac{2A}{L}\,x & 0 \le x \le \dfrac{L}{2} \\ \dfrac{2A}{L}\,(L-x) & \dfrac{L}{2} \le x \le L \end{cases} \qquad g(x) = 0$$

Calculate the energy of the *fundamental harmonic*, that is, E_1.

e) This energy corresponds to the intensity of the sound, or volume, produced by the vibration of the string. Study its dependance upon the parameters of the problem: density, length, tension, and initial stretching.

Problem 4.1.8

a) Show that given any solution of the previous Cauchy-Dirichlet problem associated to the wave equation, its total energy,

$$E(t) = \frac{\rho}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{T_0}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx,$$

is constant with time. To do this, multiply the wave equation by $\dfrac{\partial u}{\partial t}$, integrate over $[0, L]$ and deduce that $E'(t) = 0$.

b) Show that this result implies uniqueness of the solution to the problem, since the energy should be equal to the initial energy.

Problem 4.1.9 Solve the following problem for the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \ , t > 0, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi, t) = 0, & t > 0, \\ u(x,0) = 0, & 0 < x < \pi, \\ \dfrac{\partial u}{\partial t}(x,0) = 8\sin^2 x, & 0 < x < \pi. \end{cases}$$

Problem 4.1.10 Find the solution of the problem for the damped wave equation ($c > 0$):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} - \alpha \dfrac{\partial u}{\partial t}, & 0 < x < \pi, \ t > 0, \\ u(0,t) = u(\pi, t) = 0, & t > 0, \\ u(x,0) = \sin 2x, & 0 < x < \pi, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Problem 4.1.11

a) Show by separation of variables in polar coordinates, that the solution of the Laplace equation over a disk, $D = \{0 < r < a, \ -\pi \le \theta \le \pi\}$,

$$\begin{cases} \Delta u = 0, & D, \\ u = f, & \partial D, \end{cases}$$

is

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right] d\phi.$$

b) Using the identity $\cos z = \mathrm{Re}(e^{iz})$, add the resulting geometric series in order to obtain *Poisson's Integral Formula*.

c) Calculate the value of u at the origin, obtaining what is known as *Mean value theorem for harmonic functions.*

Problem 4.1.12 Solve the Laplace equation over the unit disk with boundary condition $u(1,\theta) = \sin^3\theta$.

Problem 4.1.13 Solve the Laplace equation over the semicircle $\{0 < r < a,\ 0 < \theta < \pi\}$, with the following boundary conditions:

a) $u = 0$ over the diameter and $u(a,\theta) = g(\theta)$.

b) The diameter is isolated and $u(a,\theta) = g(\theta)$.

Problem 4.1.14 Solve the Laplace equation over a 60° wedge, $\{0 < r < a,\ 0 < \theta < \pi/3\}$, with the following boundary conditions:

$$u(r,0) = 0, \quad u(r,\pi/3) = 0, \quad u(a,\theta) = f(\theta).$$

Problem 4.1.15 Solve the Laplace equation over a circular ring $\{a < r < b\}$, with the following boundary conditions:

$$i)\ \ u(a,\theta) = f(\theta), \quad u(b,\theta) = 0; \qquad ii)\ \ u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta);$$

$$iii)\ \ \frac{\partial u}{\partial r}(a,\theta) = 0, \quad u(b,\theta) = g(\theta); \qquad iv)\ \ \frac{\partial u}{\partial r}(a,\theta) = f(\theta), \quad \frac{\partial u}{\partial r}(b,\theta) = 0.$$

If it is necessary to impose any solvability condition, you should state and explain it in physical terms.

Problem 4.1.16 Solve the Laplace equation over a quarter of a ring $\{a < r < b,\ 0 < \theta < \pi/2\}$, with the following boundary conditions:

$$u(r,0) = u(r,\pi/2) = 0, \quad u(a,\theta) = 0, \quad u(b,\theta) = f(\theta).$$

4.2 Fourier series

Problem 4.2.1 Draw the Fourier series for $f(x)$ over the interval $[-L, L]$ for the following functions and compare with the function $f(x)$:

$$i)\ \ f(x) = x^2; \qquad ii)\ \ f(x) = e^x;$$

$$iii)\ \ f(x) = \begin{cases} 0, & x < 0, \\ x+1, & x \geq 0. \end{cases}$$

Problem 4.2.2 Determine the sine Fourier series for the following functions:

$$i)\ \ f(x) = \pi - x,\ \ 0 \leq x \leq \pi; \qquad ii)\ \ f(x) = \begin{cases} 1, & 0 \leq x \leq \pi/2, \\ 2, & \pi/2 < x \leq \pi; \end{cases}$$

$$iii)\ \ f(x) = x^2,\ \ 0 \leq x \leq \pi; \qquad iv)\ \ f(x) = \cos x,\ \ 0 \leq x \leq \pi.$$

Problem 4.2.3 Let $f(x)$ be an even function about $x = L/2$.

a) Show that the odd coefficients (n odd) of a cosine Fourier series over the interval $[0, L]$ are zero.

b) Obtain the cosine Fourier series of $f(x)$ over the interval $[0, L/2]$.

Problem 4.2.4 Let f be an odd function with period equal to 2 such that $f(x) = 1 - x$ if $0 \leq x \leq 1$. Is it possible to differentiate term by term the Fourier sine series for $f(x)$ and obtain the Fourier cosine series of $f'(x)$? Explain your answer.

Problem 4.2.5 In this problem, we try to obtain the coefficients of the Fourier cosine series for e^x. Find the mistakes in the following argument:

Consider the Fourier cosine series $\mathrm{e}^x = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$. Derive twice and obtain

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}.$$

Setting both cosine Fourier series equal to each other we deduce that $A_n = 0$ for every $n \geq 0$, a result which is *obviously incorrect*.
Correct the mistakes and obtain A_n.

Problem 4.2.6 Obtain the Fourier sine series of the function $\cosh x$ on $[0, L]$ deriving twice the series.

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