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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Problems

Degree in Biomedical Engineering

Chapter 5

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5 Sturm-Liouville problems

5.1 Eigenvalues and eigenfunctions

Problem 5.1.1 Determine the eigenvalues and eigenfunctions of the Sturm-Liouville problems corresponding to the equation $\varphi'' + \lambda\varphi = 0$ over the interval $0 < x < 1$, with the following boundary conditions:

$$i)\quad \varphi(0) = \varphi(1) = 0, \qquad ii)\quad \varphi'(0) = \varphi'(1) = 0,$$

$$iii)\quad \varphi(0) = \varphi'(1) = 0.$$

Problem 5.1.2 Determine the eigenvalues and eigenfunctions of the following problems:

$$i)\quad y'' + y' + (1+\lambda)y = 0, \qquad y(0) = y(1) = 0;$$

$$ii)\quad y'' - 3y' + 3(1+\lambda)y = 0, \qquad y'(0) = y'(1) = 0;$$

$$iii)\quad y'' + 2y' + (1-\lambda)y = 0, \qquad y(0) = y'(1) = 0.$$

Problem 5.1.3 Show that the eigenvalues of the problem

$$\begin{cases} \phi'' + (\lambda - x^2)\phi = 0, & 0 < x < 1, \\ \phi'(0) = \phi'(1) = 0, \end{cases}$$

are strictly positive.

Problem 5.1.4 Use the Rayleigh quotient to obtain a reasonably precise upper bound for the first eigenvalue for the following problems:

$$i)\quad \varphi'' + (\lambda - x^2)\varphi = 0, \qquad \varphi'(0) = \varphi(1) = 0;$$

$$ii)\quad \varphi'' + (\lambda - x)\varphi = 0, \qquad \varphi'(0) = 2\varphi(1) + \varphi'(1) = 0;$$

$$iii)\quad \varphi'' + \lambda\varphi = 0, \qquad \varphi(0) = \varphi(1) + \varphi'(1) = 0.$$

5.2 Generalized Fourier series

Problem 5.2.1 Consider the heat flow problem with convection

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} - V_0\dfrac{\partial u}{\partial x}, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L. \end{cases}$$

$a)$ Show that the spatial ODE obtained by separation of variables is not Sturm-Liouville type.

$b)$ Solve the problem.

Problem 5.2.2 Consider the ODE

$$\phi'' + \alpha(x)\,\phi' + [\lambda\,\beta(x) + \gamma(x)]\,\phi = 0.$$

Determine the factor $H(x)$ by which we should multiply the equation in order to reduce it to a Sturm-Liouville form:

$$\left[p(x)\,\phi'\right]' + \left[\lambda\,\sigma(x) + q(x)\right]\phi = 0.$$

SOLUTION: We must have $H = p$, $\alpha H = p'$, that implies $H(x) = e^{\int^x \alpha(s)ds}$.

Problem 5.2.3 Apply the previous problem to

$$\begin{cases} x^2\,\phi'' + x\,\phi' + \lambda\,\phi = 0, \quad 1 < x < b, \\ \phi(1) = \phi(b) = 0. \end{cases}$$

a) What is the integrating factor?

b) Show that $\lambda \geq 0$.

c) Given that the ODE is equidimensional (Euler), determine all the positive eigenvalues. Is $\lambda = 0$ an eigenvalue?

d) Are the eigenfunctions orthogonal? For which weight function? Verify the orthogonality by performing the corresponding integrals.

e) Verify that the nth eigenfunction has $n-1$ zeros on the interval $(1, b)$.

Problem 5.2.4 Solve the problem for the radial heat equation over a disk

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right), & 0 < r < a,\ t > 0, \\ u(a,t) = 0, & t > 0, \\ u(r,0) = f(r), & 0 < r < a. \end{cases}$$

Problem 5.2.5 Consider the PDE associated with the one-dimensional wave operator

$$\rho\,\frac{\partial^2 u}{\partial t^2} = T_0\,\frac{\partial^2 u}{\partial x^2} + \alpha u + \beta\frac{\partial u}{\partial t}.$$

a) Give a brief physical interpretation. What signs should α and β have according to this interpretation?

b) Suppose that ρ, α, β are functions of x. Show that the method of separation of variables is applicable only if $\beta = c\rho$, where c is a constant.

c) If $\beta = c\rho$, show that the spatial equation is Sturm-Liouville. Solve the time equation.

Problem 5.2.6 Solve the telegraph problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + a\dfrac{\partial u}{\partial t} + bu = 0, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L, \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & 0 < x < L. \end{cases}$$

Problem 5.2.7 Consider the eigenvalue problem over a rectangle

$$\begin{cases} \Delta\phi + \lambda\phi = 0, & 0 < x < L,\ 0 < y < H, \\ \phi(x,0) = \phi(x,H) = 0, & 0 < x < L, \\ \dfrac{\partial\phi}{\partial x}(0,y) = \dfrac{\partial\phi}{\partial x}(L,y) = 0, & 0 < y < H. \end{cases}$$

a) Decompose it into two one-dimensional eigenvalue problems and calculate the eigenvalues and eigenfunctions.

b) Show that most of the eigenvalues are associated with more than one linearly independent eigenfunctions in the cases $L = H$ and $L = 2H$.

c) Show that the eigenfunctions are orthogonal over the given two-dimensional region using two one-dimensional orthogonality relations.

Problem 5.2.8 The vertical displacement of a non-uniform membrane $\Omega \subset \mathbb{R}^2$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

where $c = c(x,y)$, and the condition $u = 0$ is imposed over the membrane's boundary, which has an irregular form. When variables are separate, we obtain an eigenvalue problem for the laplacian over Ω.

a) Prove that the eigenfunctions corresponding to different eigenvalues are orthogonal. For which weight?

b) Prove that $\lambda \geq 0$. Can $\lambda = 0$ be an eigenvalue?

c) Assuming that these eigenvalues are known, determine the frequencies of vibration.

Problem 5.2.9 Consider a vibrating membrane occupying a right circular sector, $\{0 < r < a,\ 0 < \theta < \pi/2\}$, with $u = 0$ over all of its boundary. and let $c > 0$ be the propagation velocity.

a) Determine the frequencies of vibration.

b) Solve the initial value problem if

$$u(r,\theta,0) = g(r,\theta), \qquad \frac{\partial u}{\partial t}(r,\theta,0) = 0.$$

Problem 5.2.10 Solve the heat equation $\dfrac{\partial u}{\partial t} = k\,\Delta u$ over a cylinder of radius a and height H, with the initial condition $u(r,\theta,z,0) = f(r,z)$ (independent of θ), if the boundary conditions are

$$u(r,\theta,0,t) = u(r,\theta,H,t) = 0, \qquad u(a,\theta,z,t) = 0.$$

Problem 5.2.11 Solve the heat equation over the quarter of a cylinder $\{0 < \theta < \pi/2,\ 0 < r < a,\ 0 < z < H\}$, with the initial condition $u(r,\theta,z,0) = f(r,\theta,z)$, if the boundary has homogeneous Neumann conditions, that is,

$$\frac{\partial u}{\partial z}(r,\theta,0,t) = \frac{\partial u}{\partial z}(r,\theta,H,t) = 0,$$

$$\frac{\partial u}{\partial \theta}(r,0,z,t) = \frac{\partial u}{\partial \theta}(r,\pi/2,z,t) = 0,$$

$$\frac{\partial u}{\partial r}(a,\theta,z,t) = 0.$$

Explain what temperature distribution is expected when $t \to \infty$.

– $\mathcal{ERC}$–