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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Problems

Degree in Biomedical Engineering

Chapter 6

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6 Fourier transform

6.1 Basic properties

Problem 6.1.1

a) Show that if $\int_{-\infty}^{\infty} f(x)\,dx = 1$, then the re-scaled function $g(x) = \alpha\, f(\alpha x)$, $\alpha > 0$, also has integral one.

b) Calculate in that case the Fourier transform of g in terms of the Fourier transform of f and study its behaviour as α changes: The "wide" functions have Fourier transforms with a narrow peak near $\omega = 0$ and the "narrow" functions have a wide Fourier transform near $\omega = 0$.

c) For $\int_{\mathbb{R}^n} f(\vec{x})\,d\vec{x} = 1$, consider the re-scaled function $g(\vec{x}) = \beta\, f(\alpha\vec{x})$ similarly to part *a*) and prove that g also has integral one.

d) Compute $\widehat{g}$ in terms of $\widehat{f}$.

Problem 6.1.2

a) Given $x_0 \in \mathbb{R}$, $\beta > 0$, for what values of α has unit area the function $g(x) = \alpha\, e^{-\beta(x-x_0)^2}$ defined for $x \in \mathbb{R}$?

b) Show that for that value of α we have $\lim_{\beta\to\infty} g(x) = 0$ for every $x \neq x_0$.

c) Compute in that case the Fourier transform of $g(x)$ and obtain the limit as $\beta \to \infty$.

d) Use the integration properties of $\delta(x - x_0)$ to obtain its Fourier transform.

e) Show that the Gauss kernel is an approximation to the Dirac delta when $t \to 0$.

Problem 6.1.3 Find the Fourier transform of the function $f(x) = \begin{cases} 1, & -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$

Problem 6.1.4 Compute the Fourier transform of the function $f(x) = \mathrm{e}^{-\alpha|x|}$ and check graphically what is said in problem 6.1.1.*b*).

Problem 6.1.5 Compute the Fourier transform of Δf in terms of the Fourier transform of f.
f.

Problem 6.1.6 Given a function f, we define $g = f * h$, where $h(x) = f(-x)$. Show, writing $g(0)$ in two different ways, that $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$.

Problem 6.1.7 Given a function f, we consider $u = g * f$, where $\widehat{g}(\omega) = \dfrac{1}{\omega^2+1}$. Compute $-u'' + u$.

6.2 Resolution of equations

Problem 6.2.1 For a fixed function f and a real number $\lambda > 0$, solve the equation $-u'' + \lambda^2 u = f$ using Fourier transform.

Problem 6.2.2

a) Use the properties of the convolution and the shift to solve with Fourier transform the diffusion equation with convection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2} + c\,\dfrac{\partial u}{\partial x}, & x \in \mathbb{R},\ \ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where $k > 0$, $c \in \mathbb{R}$.

b) Consider $f(x) = \delta(x)$. Draw the solution corresponding to different values of $t > 0$. Discuss the meaning of the convection $c\,\partial u/\partial x$.

c) Does the solution suggest a change of variables to simplify the equation?

Problem 6.2.3

a) Solve the diffusion equation with absorption:

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2} - \gamma u, & x \in \mathbb{R},\ \ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases}$$

where k, $\gamma > 0$.

b) Does the solution suggest a change of variables to simplify the equation?

Problem 6.2.4

a) Compute for fixed $a > 0$ the Fourier transform of the function $f(x) = (a - |x|)_+$.

b) Solve using Fourier transform the problem for the wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = \delta(x), & x \in \mathbb{R},\ \ t > 0, \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Problem 6.2.5 Solve the following Laplace equation over an infinite strip, using the Fourier transform in the y variable:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < L,\ y \in \mathbb{R}, \\ u(0,y) = g_1(y), & y \in \mathbb{R}, \\ u(L,y) = g_2(y), & y \in \mathbb{R}. \end{cases}$$