uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 1

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http://ocw.uc3m.es/matematicas

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1 First order differential equations

1.1 Elementary resolution methods

Problem 1.1.1 Here $k > 0$ denotes a constant. *a*) $\frac{dN}{dt} = -kN$. *b*) $\frac{dP}{dt} = kP(P_0 - P)$. *c*) $\frac{dP}{dT} = k\frac{P}{T^2}$. *d*) $\frac{d^2y}{dt^2} = \frac{F}{m}$.

Problem 1.1.2 *a*) Solving the system $\begin{cases} y = cx + x^2 \\ y' = c + 2x \end{cases}$ to eliminate c, one obtains $xy' - y - x^2 = 0$; *b*) $(y')^2 + xy' - y = 0$; *c*) the family is $(x-a)^2 + y^2 = r^2$, with $a, r \in \mathbb{R}$; since there are two constants we derive twice, $yy'' + (y')^2 + 1 = 0$; *d*) $y' = y^2 - 1$; *e*) $\frac{r'}{r} = \frac{\sin\theta}{1 - \cos\theta}$.

Problem 1.1.3 *i*) $e^{2y}dy = e^{3x}dx \rightsquigarrow y = \frac{1}{2}\log(\frac{2}{3}e^{3x} + c)$, *ii*) $\frac{e^y}{y+1} = Cxe^x$, *iii*) $(1-y)e^y = e^{-x} + \frac{1}{3}e^{-3x} + c$, *iv*) $y^2 + 2 = c(x^2 + 4)$, *v*) $y - \frac{1}{y} = \arctan x + c$, *vi*) $y^2 = x^2 + x + c$, *vii*) $e^{y-x} = \frac{C(y+3)^5}{(x+4)^5}$, *viii*) $e^{y-x} = \frac{C(x-3)^5}{(y-1)^2}$, *ix*) $y = \arctan(ce^{2\sin x})$, *x*) $y = ce^{x - x^2/2}$.

Problem 1.1.4 *i*) Writing $z = \frac{y}{x}$ we have $z' = \frac{1}{x}(\frac{2z^2 - 1}{z} - z)$, that implies $\frac{x^2}{z^2 - 1} = k$, that is $y^2 = x^2 + cx^4$, *ii*) $\frac{z}{1+z} = cx^2 \rightsquigarrow y = \frac{x^3}{c - x^2}$, *iii*) $\operatorname{arctg} z = cx^3 \rightsquigarrow y = x\tan(cx^3)$, *iv*) $-\cos z = \log|x| + k \rightsquigarrow y = x\arccos(\log(\frac{c}{|x|}))$, *v*) $e^z = 2\log|x| + k \rightsquigarrow y = x\log(\log(cx^2))$, *vi*) $\log|z| - \frac{2}{\sqrt{z}} = -\log|x| + k \rightsquigarrow \log|y| = 2\sqrt{\frac{x}{y}} + k$, *vii*) $\log|z^4 + z| = \log|x| + k \rightsquigarrow y^4 + x^3y = cx^5$, *viii*) $\frac{z^2}{2} = \log|x| + k \rightsquigarrow y^2 = x^2\log cx^2$, *ix*) $\frac{1}{2}e^{2/z} - 4\log|z| = 4\log|x| + k \rightsquigarrow e^{2x/y} = \log cy^8$, *x*) $\log|\sec z| = \log|x| + k \rightsquigarrow y = x\arccos(\frac{c}{x})$.

Problem 1.1.5

a) If (x_0, y_0) is the intersection point of the two straight lines $Ax + By + C = 0$, $Dx + Ey + F = 0$, in the new variables the lines pass through the origin, $Aw + Bz = 0$, $Dw + Ez = 0$. The equation becomes
$$z' = f\left(\frac{Aw + Bz}{Dw + Ez}\right)$$
that is homogeneous.

b) Substituting $D = AE/B$,
$$\frac{Ax + By + C}{Dx + Ey + F} = \frac{Ax + By + C}{\frac{E}{B}(Ax + By) + F)},$$
that is a function of $Ax + By$, so if we define $z = Ax + By$ the equation is
$$z' = A + By' = A + Bf\left(\frac{z + C}{\frac{E}{B}z + F}\right).$$

Problem 1.1.6
SOLUTION: *i)* $z = 2x + 2y + 4 \rightsquigarrow z' = 2(1 + z^2) \rightsquigarrow z = \mathrm{tg}(2x + c) \rightsquigarrow y = -2 - x + \frac{1}{2}\mathrm{tg}(2x + c)$; *ii)* $z = x - y + 5 \rightsquigarrow \mathrm{cotg}(x - y + 5) + \mathrm{cosec}(x - y + 5) = c - x$; *iii)* $w = x - 1$, $z = y + 5 \rightsquigarrow 2\arctan\frac{y + 5}{x - 1} = \log c((x - 1)^2 + (y + 5)^2)$; *iv)* $z = x + y \rightsquigarrow x - y + 5\log|x + y - 1| = c$.

Problem 1.1.7 *i)*, *iii)* y *vi)* are not exact, *ii)* If ϕ is the potential of the vector field $F = (y - x^3, x + y^3)$, then we have $\frac{\partial\phi}{\partial x} = y - x^3 \rightsquigarrow \phi = xy - \frac{x^4}{4} + g(y)$; $\frac{\partial\phi}{\partial y} = x + g' = x + y^3 \rightsquigarrow g = \frac{y^4}{4}$; so $\phi = c \rightsquigarrow y^4 - x^4 + 4xy = c$, *iv)* $x^2y^2|x^2 - 4y^2| = c$, *v)* $xe^y + \sin x\cos y = c$, *vii)* $x^2y^3 + y\sin x = c$, *viii)* $e^{2x+c} = \frac{1 + xy}{1 - xy}$, *ix)* $x^2y^2 + 2x^3y = c$.

Problem 1.1.8 *a)* $\frac{\partial M}{\partial y} = f' = \frac{\partial M}{\partial x} = -1 \rightsquigarrow f(y) = c - y$; *b)* $(\mu f)_y = -\mu \rightsquigarrow \mu = \frac{1}{f}e^{(} - \int\frac{1}{f})$; *c)* $\mu(y) = \frac{1}{y^2} \rightsquigarrow x = y(c + \log|y|)$.

Problem 1.1.9 *i)* We try $\mu = \mu(x)$, $\frac{\partial((4x + 3y^3)\mu)}{\partial y} = \frac{\partial(3xy^2\mu)}{\partial x} \rightsquigarrow 9y^2\mu = 3y^2\mu + 3xy^2\mu' \rightsquigarrow \frac{\mu'}{\mu} = \frac{2}{x} \rightsquigarrow \mu = x^2$; the solution is $x^4 + x^3y^3 = c$; *ii)* if we try again $\mu = \mu(x)$, we arrive to the equation $\frac{\mu'}{\mu} = \frac{8xy + 2}{6y^3 - x}$, that is impossible because the right hand term depends only on x; then we try $\mu = \mu(y)$, and obtain $\mu(y) = \frac{1}{y^2}$, and the final solution $2x^2 + \frac{x}{y} + 3y^2 = c$; *iii)* $\mu(y) = \sin y$, $x^2\sin y = c$; *iv)* $\mu(y) = \frac{1}{y^2}$, $\frac{x^2}{y} + x = c$; *v)* $\mu(y) = y^3$, $y^5\sin x + y^4 = c$; *vi)* $\mu(x) = x^2$, $x^6y = c$.

Problem 1.1.10

$$\frac{\partial(7x^{4+n}y^{1+m} - 3x^ny^{8+m})}{\partial y} = \frac{\partial(2x^{5+n}y^m - 9x^{1+n}y^{7+m})}{\partial x} \rightsquigarrow \begin{cases} 7(m+1) = 2(n+5) \\ 3(m+8) = 9(n+1) \end{cases},$$

that implies $n = 2$, $m = 1$; final solution $x^7y^2 - x^3y^9 = c$.

Problem 1.1.11 *i)* $\mu = e^{-\int\frac{3}{x}} = \frac{1}{x^3} \rightsquigarrow y = \frac{1}{\mu}(\int x^3\mu + c) = x^3(x + c)$, *ii)* $\mu = e^x \rightsquigarrow y = e^{-x}(c + \arctan e^x)$, *iii)* $\mu = 1 + x^2 \rightsquigarrow y = \frac{\log|\sin x| + c}{1 + x^2}$, *iv)* $\mu = e^x \rightsquigarrow y = (x^2 + c)e^{-x} + x^2 - 2x + 2$, *v)* $\mu = \sin x \rightsquigarrow y = (x^2 + c)\,\mathrm{cosec}\,x$, *vi)* $\mu = \frac{1}{x^2} \rightsquigarrow y = x^2(c - x)$, *vii)* $\mu = x\sin x \rightsquigarrow y = \frac{\sin x - x\cos x + c}{x\sin x}$, *viii)* $\mu = e^{-y^2} \rightsquigarrow x = (3y^2 + c)e^{y^2}$.

Problem 1.1.12 *i)* $z = \frac{1}{y} \rightsquigarrow z' = \frac{z}{x} - 1 \rightsquigarrow z = x(c - \log|x|) \rightsquigarrow y = \frac{1}{x(c - \log|x|)}$; *ii)* $z = \frac{1}{y^2} \rightsquigarrow z' = \frac{2z}{x} - 2x^3 \rightsquigarrow z = x^2(c - x^2) \rightsquigarrow x^2y^2(c - x^2) = 1$; *iii)* $z = y^3 \rightsquigarrow x^3y^3 + 3x(6 - x^2)\sin x + 9(2 - x^2)\cos x = c$; *iv)* $z = y^{1/3} \rightsquigarrow y = (2x^4 + cx^2)^3$; *v)* $z = \frac{1}{y^2} \rightsquigarrow y^2(\frac{1}{20} - \frac{x}{2} + ce^{-10x}) = 1$.

Problem 1.1.13 This is a Bernoulli type equation, that can be solved with the change $z = \dfrac{1}{y^2}$; it is also a homogeneous equation, that can be solved with the change $w = \dfrac{y}{x}$; the solution is $x^4 - x^2y^2 + cy^2 = 0$.

Problem 1.1.14 $z = \log y \rightsquigarrow z' = Qz - P$, which is linear; the solution is $y = \mathrm{e}^{cx+2x^2}$.

Problem 1.1.15 If $\mu(x, y) = \phi(s)$, $s = x + y^2$, we have

$$\begin{aligned}
&\frac{\partial((3x + 2y + y^2)\mu)}{\partial y} = \frac{\partial((x + 4xy + 5y^2)\mu)}{\partial x} \\
&\rightsquigarrow (2 + 2y)\phi + (3x + 2y + y^2)2y\phi' = (1 + 4y)\phi + (x + 4xy + 5y^2)\phi' \\
&\rightsquigarrow \frac{\phi'}{\phi} = \frac{1}{x + y^2} = \frac{1}{s} \rightsquigarrow \phi = s;
\end{aligned}$$

the equation $(x + y^2)(3x + 2y + y^2)\,dx + (x + y^2)(x + 4xy + 5y^2)\,dy = 0$ is exact with solution $x^3 + x^2y + 2x^2y^2 + 2xy^3 + xy^4 + y^5 = c$.

Problem 1.1.16 *i)* Homogeneous and also Bernoulli type with $n = 2$, $y^3 = x^3(3\log x + c)$, *ii)* linear, $2xy + \cos 2x = c$, *iii)* exact differential, $3x^2 + 2y^2 + x^2y^2 = c$, *iv)* exact differential, $\mathrm{e}^{xy} + \sin y = c$, *v)* Bernoulli type and also exact differential, $xy^3 + x\cos x - \sin x = c$, *vi)* Bernoulli type equation or we can use an integrating factor, $x^3y^2 + \dfrac{2}{3}x^3 - 2\log x = c$.

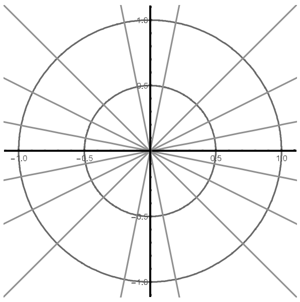
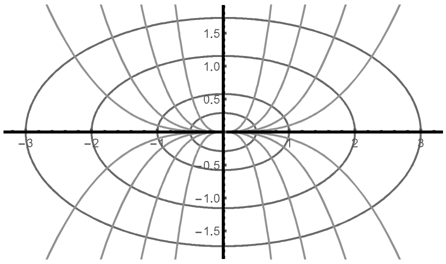
Problem 1.1.17 *i)* $p = y' \rightsquigarrow p' = -\dfrac{p}{x} \rightsquigarrow p = \dfrac{c}{x} \rightsquigarrow y = c_1 \log|x| + c_2$; *ii)* $p(y) = y'(x) \rightsquigarrow p' = \dfrac{p}{y} \rightsquigarrow p = cy \rightsquigarrow y = c_2\mathrm{e}^{c_1x}$; *iii)* $p(y) = y'(x) \rightsquigarrow p' = -\dfrac{p}{y} \rightsquigarrow p = \dfrac{c}{y} \rightsquigarrow y^2 = c_1x + c_2$; *iv)* $p = y' \rightsquigarrow p' = \dfrac{p}{x} + \dfrac{p^3}{x} \rightsquigarrow p = \dfrac{x}{\sqrt{c - x^2}} \rightsquigarrow x^2 + (y - c_2)^2 = c_1^2$; *v)* $p(y) = y'(x) \rightsquigarrow p' = \dfrac{p}{y} \rightsquigarrow p = cy \rightsquigarrow p' = \dfrac{k^2y}{p} \rightsquigarrow p = \pm\sqrt{k^2y^2 + c} \rightsquigarrow y = c_1\mathrm{e}^{kx} + c_2\mathrm{e}^{-kx}$; *vi)* $p = y' \rightsquigarrow p' = -\dfrac{p}{x} + 4 \rightsquigarrow p = 2x + \dfrac{c}{x} \rightsquigarrow y = x^2 + c_1\log|x| + c_2$.

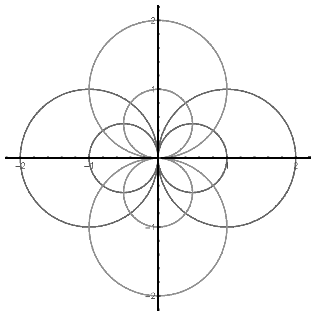
Problem 1.1.18 $z = y - x$ verifies $z' = z^{-2}$, that is of separate variables, $z = \dfrac{-1}{x + c}$; the solution is $y(x) = x - \dfrac{1}{x + c}$.

Problem 1.1.19 Substituting $y = kx^\alpha$ we have $k\alpha x^{\alpha-1} = 3x^6 - 2kx^{\alpha+4} - \dfrac{1}{3}k^2x^{2\alpha+2} + 2kx^{\alpha-1}$, so $\alpha = 2$, $k = 3$; writing $z = y - 3x^2$ we obtain the equation $z' = \dfrac{x^2z^2}{3} + \dfrac{2z}{x}$; now $w = z^{-1}$ verifies the linear equation $w' = -\dfrac{x^2}{3} - \dfrac{2w}{x}$, that has the solution $w = \dfrac{1}{x^2}\left(c - \dfrac{1}{3}\displaystyle\int x^4\right) = \dfrac{15c - x^5}{15x^2}$; finally, $y = 3x^2 + \dfrac{15x^2}{k - x^5}$.

1.2 Applications

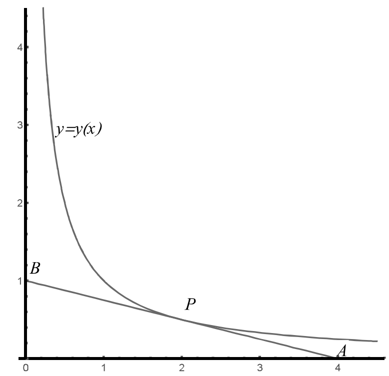
Problem 1.2.1 *i)* $y_1' = \dfrac{y}{x} = -\dfrac{1}{y_2'}$, *ii)* $y_1' = \dfrac{3y}{x} = -\dfrac{1}{y_2'}$, *iii)* $y_1' = \dfrac{y^2 - x^2}{2xy} = -\dfrac{1}{y_2'}$.

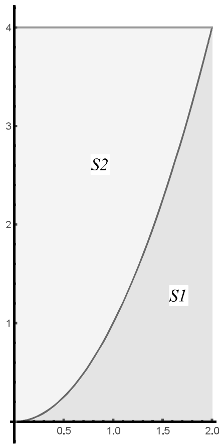


Problem 1.2.2 *i*) Family equation, $y' = \dfrac{2y}{x} \rightsquigarrow$, orthogonal family equation, $y' = \dfrac{-x}{2y}$, solution $2y^2 + x^2 = k$; *ii*) $y' = \dfrac{xy}{1-y^2} \rightsquigarrow 2\log|y| = x^2 + y^2 + k$; *iii*) $y' = \dfrac{1}{y} \rightsquigarrow y^2 = 2x + k$; *iv*) $y' = \dfrac{-2x}{3y} \rightsquigarrow 2x^2 + 3y^2 = k$; *v*) $y' = \dfrac{-x}{\operatorname{tgh} y} \rightsquigarrow 2\log\cosh y + x^2 = k$; *vi*) $y' = \dfrac{-x^2}{y^2} \rightsquigarrow x^3 + y^3 = k$; *vii*) $y' = \dfrac{x}{y^2} \rightsquigarrow 2y^3 - 3x^2 = k$; *viii*) $y' = \dfrac{x^{2/3}}{y^{2/3}} \rightsquigarrow y^{5/3} - x^{5/3} = k$; *ix*) $y' = \dfrac{x}{y} \rightsquigarrow x^2 - y^2 = k$.

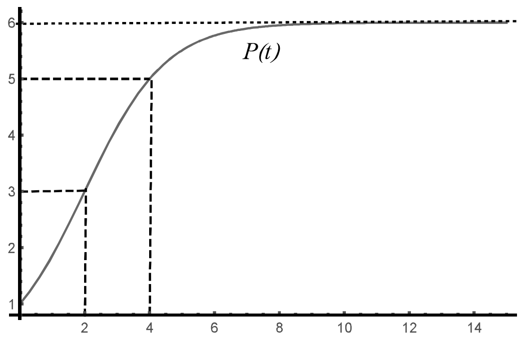
Problem 1.2.3 For a curve $y(x)$, build the tangent line at $P = (x_0, y_0)$, $y = y(x_0) + y'(x_0)(x - x_0)$; the intersection with the axes are, $A = ((x_0 - \dfrac{y(x_0)}{y'(x_0)}, 0)$, $B = (0, y(x_0) - x_0 y'(x_0))$; making equal the two distances $|\overline{AP}|^2 = |\overline{BP}|^2 \rightsquigarrow x_0^2 + x_0^2 y'(x_0)^2 = \dfrac{y(x_0)^2}{y'(x_0)^2} + y(x_0)^2$; since P is a generic point, we have the ODE $x^2(1 + (y')^2) = \dfrac{y^2}{(y')^2}(1 + (y')^2) \rightsquigarrow y' = \pm\dfrac{y}{x} \rightsquigarrow y = \dfrac{c}{x}$, $y = cx$ (the second is trivial).



Problem 1.2.4 If the lower part, S_1 has half the area of the upper part, S_2, then it is a third of the area of the rectangle, that is, $\displaystyle\int_0^x y(s)\,ds = \frac{1}{3}xy \rightsquigarrow 3y = xy' + y \rightsquigarrow y = cx^2$; if the proportion is on the contrary we obtain $y = c\sqrt{x}$.



Problem 1.2.5 *a)* This a separate variables equation, $P(t) = \dfrac{1}{\frac{b}{a} + c\mathrm{e}^{-at}}$, using the initial condition, $P(t) = \dfrac{1}{\frac{b}{a} + (\frac{1}{M} - \frac{b}{a})\mathrm{e}^{-at}}$; *b)* if $t = 0$ corresponds to 1990 and we compute in thousands of fish we have $M = 1$, $P(2) = 3$, $P(4) = 5$, that implies the system $\begin{cases} \dfrac{1}{\frac{b}{a} + (1 - \frac{b}{a})\mathrm{e}^{-2a}} = 3, \\ \dfrac{1}{\frac{b}{a} + (1 - \frac{b}{a})\mathrm{e}^{-4a}} = 5; \end{cases}$ denoting $\dfrac{b}{a} = X$, $\mathrm{e}^{-2a} = Y$, we have $\begin{cases} X + (1 - X)Y = \dfrac{1}{3}, \\ X + (1 - X)Y^2 = \dfrac{1}{5}, \end{cases}$ that gives $X = \dfrac{1}{6}$, $Y = \dfrac{1}{5}$; finally, since $\mathrm{e}^{-at} = Y^{t/2}$, we arrive to the solution $P(t) = \dfrac{6}{1 + 5^{1-t/2}}$; in 1998 we have $P(8) = \dfrac{125}{21} \approx$ 5.952 fish; *c)* $\lim\limits_{t\to\infty} P(t) = \dfrac{a}{b} = 6 \rightsquigarrow$ 6.000 fish.



Problem 1.2.6 *a)* Volume $V = \dfrac{4}{3}\pi R^3$, area $A = 4\pi R^2$; the equation $V' = -kA$ implies $4\pi R^2 R' = -k4\pi R^2$, that is, $R' = -k$, with solution $R(t) = c - kt$; if $R(0) = 2$ and $R(\dfrac{1}{2}) = \dfrac{3}{2}$, we obtain $c = 2$, $k = 1$, which gives $R(t) = 2 - t$; $R(t) = 1$ implies $t = 1$ hour; *b)* $R(t) = 0$ implies $t = 2$ hours.

Problem 1.2.7 $d'(t) = -kd(t) \rightsquigarrow d(t) = d_0\mathrm{e}^{-kt}$, $d(\sigma) = \dfrac{d_0}{2} \rightsquigarrow \mathrm{e}^{-k} = 2^{-\frac{1}{\sigma}} \rightsquigarrow d(t) = d_0 2^{-\frac{t}{\sigma}} \rightsquigarrow$ $t = \sigma \log_2(\dfrac{d_0}{d(t)}) \rightsquigarrow T = 5600 \log_2(\dfrac{15{,}30}{d(t)}) + 67$ (Nowadays 2018);

a) $T_1 = 3.391$ years; *b*) $T_2 = 7.084$ years; *c*) $T_3 = 10.570$ years.

– $\mathcal{ERC}$–