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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 2

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2 Linear equations of higher order

2.1 Linear equations of order n with constant coefficients

Problem 2.1.1 *i)* $\lambda^2 + 2\lambda - 3 = 0 \rightsquigarrow \lambda_1 = 1, \lambda_2 = -3 \rightsquigarrow y = c_1 e^x + c_2 e^{-3x}$; *ii)* $\lambda = \pm 2 \rightsquigarrow y = c_1 e^{2x} + c_2 e^{-2x}$; *iii)* $\lambda_1 = 1, \lambda_2 = -5 \rightsquigarrow y = c_1 e^x + c_2 e^{-5x}$;*iv)* $\lambda = \pm 2\sqrt{2}\, i \rightsquigarrow y = c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x$; *v)* $\lambda = -2 \pm i \rightsquigarrow y = e^{-2x}(c_1 \cos x + c_2 \sin x)$; *vi)* $\lambda = \dfrac{-1 \pm i\sqrt{5}}{2} \rightsquigarrow y = e^{-x/2}(c_1 \cos(\dfrac{\sqrt{5}\,x}{2}) + c_2 \sin(\dfrac{\sqrt{5}\,x}{2}))$.

Problem 2.1.2 *i)* homogeneous solution $y_h = c_1 e^{2x} + c_2 e^{-5x}$; particular solution $y_p = A e^{4x} \rightsquigarrow A = \dfrac{1}{3}$; complete solution $y(x) = c_1 e^{2x} + c_2 e^{-5x} + \dfrac{1}{3} e^{4x}$; *ii)* particular solution $y_p = Ax^2 + Bx + C \rightsquigarrow A = 5,\ B = 4,\ C = 2$; complete solution $y(x) = e^x(c_1 \cos 2x + c_2 \sin 2x) + 5x^2 + 4x + 2$; *iii)* particular solution $y_p = Axe^{-2x} \rightsquigarrow A = -4$; complete solution $y(x) = c_1 e^{3x} + e^{-2x}(c_2 - 4x)$; *iv)* particular solution $y_p = A \cos 2x + \sin 2x \rightsquigarrow A = \dfrac{3}{2},\ B = -\dfrac{1}{2}$; complete solution $y(x) = c_1 e^x + c_2 e^{2x} + \dfrac{1}{2}(3 \cos 2x - \sin 2x)$; *v)* particular solution $y_p = A e^{-2x} + Bx^2 e^x \rightsquigarrow A = \dfrac{1}{9},\ B = \dfrac{1}{2}$; complete solution $y(x) = e^x(c_1 + c_2 x + \dfrac{1}{2}x^2) + \dfrac{1}{9} e^{-2x}$; *vi)* particular solution $y_p = (Ax + B)\cos x + (Cx + D)\sin x \rightsquigarrow A = 0,\ B = -\dfrac{1}{2},\ C = -\dfrac{1}{2},\ D = 0$; complete solution $y(x) = c_1 e^x + c_2 e^{-x} - \dfrac{1}{2}(\cos x + x \sin x)$; *vii)* particular solution $y_p = e^x(A \cos x + B \sin x) \rightsquigarrow A = \dfrac{4}{5},\ B = \dfrac{3}{5}$; complete solution $y(x) = e^{3x}(c_1 + c_2 x) + \dfrac{1}{5} e^x(4 \cos x + 3 \sin x)$; *viii)* particular solution $y_p = A \cos x + B \sin x + Cx^2 + Dx + E \rightsquigarrow A = -1,\ B = 0,\ C = 1,\ D = -1,\ E = -2$; complete solution $y(x) = e^{-x/2}(c_1 \cos \dfrac{\sqrt{3}\,x}{2} + c_2 \sin \dfrac{\sqrt{3}\,x}{2}) - \cos x + x^2 - x - 2$.

Problem 2.1.3 *i)* $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 2 \rightsquigarrow y(x) = c_1 + c_2 e^x + c_2 e^{2x}$; *ii)* $y(x) = c_1 e^x + e^{-x/2}(c_2 \cos \dfrac{\sqrt{3}\,x}{2} + c_3 \sin \dfrac{\sqrt{3}\,x}{2})$; *iii)* $\lambda = \pm a$ double, $y(x) = e^{ax}(c_1 + c_2 x) + e^{-ax}(c_3 + c_4 x)$; *iv)* $y(x) = e^x(c_1 + c_2 x) + c_3 e^{-2x} + \dfrac{15}{2} + 5x + \dfrac{3}{2}x^2$; *v)* $y(x) = e^x(c_1 + c_2 x + \dfrac{1}{4}x^2) + c_3 \cos x + c_4 \sin x$.

Problem 2.1.4 *i)* The roots of the characteristic polynomial are $\lambda_1 = 0,\ \lambda_2 = 1$, so the characteristic equation is $\lambda^2 - \lambda = 0$, that corresponds to the differential equation $y'' - y' = 0$. *ii)* $\lambda_{1,2} = \pm 3i \rightsquigarrow \lambda^2 + 9 = 0 \rightsquigarrow y'' + 9y = 0$. *iii)* $\lambda_{1,2} = -2$ double $\rightsquigarrow y'' + 4y' + 4y = 0$. *iv)* $\lambda_1 = 2,\ \lambda_{2,3} = \pm i \rightsquigarrow y''' - 2y'' + y' - 2y = 0$. *v)* $\lambda_{1,2,3} = 1$ triple $\rightsquigarrow y''' - 3y'' + 3y' - y = 0$.

Problem 2.1.5 $y_1(x) - y_2(x) = 2e^{-3x}$ must be a solution of the homogeneous equation; the same happens with $y_1(x) - y_3(x) = e^{-3x} + xe^{-3x}$. So, $\lambda = -3$ must be a double root of the characteristic polynomial, so $a = 6,\ b = 9$. Besides, e^{-x} is a solution of the complete equation, so $f(x) = Ae^{-x}$; substituting we obtain $A = 4$. Finally the general solution is $y_g(x) = e^{-3x}(c_1 + c_2 x) + 4e^{-x}$.

Problem 2.1.6 C*a)* Just replace in the equation; *b)* the general solution is $x_g(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t} + e^{3t}$; imposing the data, $x(t) = 2e^t - e^{-2t} + e^{3t}$.

Problem 2.1.7 *i)* $x(t) = c_1 e^{-t} + c_2 e^{-3t} = \frac{1}{2}(e^{-t} - e^{-3t})$; *ii)* $x(t) = e^{-2t}(c_1 + c_2 t + c_3 t^2) = e^{2-2t}(-4t^2 + 4t - 2)$; *iii)* $x(t) = c_1 + c_2 t + c_3 t^2 + e^{-2t}(c_4 \cos t + c_5 \sin t) = \frac{t^2}{2} + t + 1$; *iv)* $x(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + c_5 e^{-3t} = -e^{-t} + \frac{t^2}{2} - t + 1$.

Problem 2.1.8 The general solution of the homogeneous equation is $y_h(x) = c_1 + c_2 e^x + c_3 x e^x$; then a particular solution of the complete equation is $y_p(x) = Ax + Bx^2 e^x + Ce^{5x}$, we obtain $A = 2$, $B = -12$, $C = 1/2$; the general solution is then $y_g(x) = y_h(x) + y_p(x)$, where the initial data imply $c_1 = 11$, $c_2 = 0$, $c_3 = 9$; finally, $y(x) = e^x(-11 + 9x - 12x^2) + \frac{1}{2}e^{5x} + 2x + 11$.

Problem 2.1.9 *i)* Independent solutions of the homogeneous equation are $y_1 = \cos 2x$, $y_2 = \sin 2x$, with wronskian $W = 2$; writing then $y_p = u \cos 2x + v \sin 2x$, we have

$$u = -\int \frac{1}{2} \sin 2x \,\mathrm{tg}\, 2x \, dx = \frac{1}{4}(\sin 2x - \log(\sec 2x + \mathrm{tg}\, 2x)),\ v = \int \frac{1}{2} \sin 2x \, dx = -\frac{1}{4} \sin 2x;$$

finally, $y_p(x) = \frac{-1}{4} \cos 2x \log(\sec 2x + \mathrm{tg}\, 2x)$; *ii)* $y_1 = e^{-x}$, $y_2 = xe^{-x}$, $W = e^{-2x}$,

$$u = -\int e^{2x} x e^{-x} e^{-x} \log x \, dx = -\frac{1}{4}x^2(2\log x - 1),\ v = \int e^{2x} e^{-x} e^{-x} \log x \, dx = x(\log x - 1),$$

$y_p(x) = x^2 e^{-x}(\frac{1}{2}\log x - \frac{3}{4})$; *iii)* $y_1 = e^{-x}$, $y_2 = e^{3x}$, $W = 4e^{2x}$,

$$y_p(x) = -e^{-x} \int 16x \, dx + e^{3x} \int 16x e^{-4x} \, dx = 2e^{-x}(8x^2 + 4x - 1);$$

also, we can look for a particular solution directly in the form $y_p = (Ax^2 + Bx)e^{-x}$; *iv)* $y_1 = e^{-x} \cos 2x$, $y_2 = e^{-x} \sin 2x$, $W = 2e^{-2x}$,

$$y_p(x) = -e^{-x} \cos 2x \int \frac{1}{2} \mathrm{tg}\, 2x \, dx + e^{-x} \sin 2x \int \frac{1}{2} \, dx = \frac{1}{4} e^{-x}(\cos 2x \log(\cos 2x) + 2x \sin 2x).$$

Problem 2.1.10 *i)* $k(k-1) - 4k + 4 = 0 \rightsquigarrow k_1 = 1,\ k_2 = 4 \rightsquigarrow y(x) = c_1 x + c_2 x^4$, *ii)* $k = \pm n \rightsquigarrow y(x) = c_1 x^n + c_2 x^{-n}$, *iii)* $k = 0$ double, $y(x) = c_1 \log x + c_2$; also this can be solved by order reduction; *iv)* $k_1 = 0,\ k_2 = 1,\ k_3 = -1 \rightsquigarrow y(x) = c_1 + c_2 x + c_3 x^{-1}$, *v)* $k_1 = 1,\ k_2 = 2,\ k_3 = -1 \rightsquigarrow y(x) = c_1 x + c_2 x^2 + c_3 x^{-1}$, *vi)* $k_1 = 1,\ k_{2,3} = \pm i \rightsquigarrow y(x) = c_1 x + c_2 \cos(\log x) + c_3 \sin(\log x)$, *vii)* $k_1 = -1$ double $\rightsquigarrow y(x) = c_1 x^{-1} + c_2 x^{-1} \log x + \frac{x^2}{3}$, *viii)* $k_1 = 1$ triple, $\rightsquigarrow y(x) = c_1 x + c_2 x \log x + c_3 x (\log x)^2 + \frac{x^4}{9}$.

Problem 2.1.11 Since $W = e^{-\int \frac{1}{x}} = \frac{1}{x}$, we have

$$y_2 = \frac{\cos x}{\sqrt{x}} \int \frac{x}{x \cos^2 x} \, dx = \frac{\cos x}{\sqrt{x}} \mathrm{tg}\, x = \frac{\sin x}{\sqrt{x}}.$$

Problem 2.1.12 *i)* Since $W = e^{\int \frac{2x}{x^2-1}} = x^2 - 1$, we have $y_2 = x \int \frac{x^2 - 1}{x^2} = x^2 + 1$; on the other hand, $y_p = -x \int (x^2 + 1) + (x^2 + 1) \int \frac{x(x^2 - 1)}{x^2 - 1} = -\frac{x^4}{3} - x^2 + \frac{x^4}{2} + \frac{x^2}{2}$; finally $y(x) =$

$c_1 x + c_2(x^2+1) + \frac{x^4}{6} - \frac{x^2}{2}$, *ii)* Since $W = \mathrm{e}^{\int \frac{x^2-2}{x^2+x}} = \frac{\mathrm{e}^x(x+1)}{x^2}$, we have $y_2 = \frac{1}{x}\int (x+1)\mathrm{e}^x = \mathrm{e}^x$; and then, $y_p = -\frac{1}{x}\int x^2 + \mathrm{e}^x \int x\mathrm{e}^{-x} = -\frac{x^2}{3} - x - 1$; finally $y(x) = c_1 x^{-1} + c_2\mathrm{e}^x - \frac{x^2}{3} - x - 1$.

2.2 Applications

Problem 2.2.1 $k_1 = 0$, $k_2 = 2$, $k_3 = -1$, $k_4 = -3 \rightsquigarrow x(t) = c_1 + c_2 t^2 + c_3 t^{-1} + c_4 t^{-3}$.

Problem 2.2.2 Since $Q' = I$, the problem for Q is

$$\begin{cases} Q'' + 2Q' + 10Q = \sin 2t, \\ Q(0) = 0, \\ Q'(0) = 0. \end{cases}$$

The solution is $Q = e^{-t}(c_1\cos 3t + c_2 \sin 3t) + A\cos 2t + B\sin 2t$, where A and B are obtained by the method of undetermined coefficients, $A = -\frac{1}{13}$, $B = \frac{3}{26}$, while c_1 and c_2 are obtained from the initial data, $c_1 = \frac{1}{13}$, $c_2 = -\frac{2}{39}$. Finally, $I = Q' = -\frac{1}{13}\mathrm{e}^{-t}(3\cos 3t + \frac{7}{3}\sin 3t) + \frac{1}{13}(3\cos 2t + 2\sin 2t)$. We could also have studied the problem for the current intensity:

$$\begin{cases} I'' + 2I' + 10I = 2\cos 2t, \\ I(0) = 0, \\ I'(0) = -10Q(0) - 2Q'(0) = 0. \end{cases}$$

Problem 2.2.3 The spring constant is $k = \frac{100Nw}{2m} = 50Nw/m$. Then the resulting problem is

$$\begin{cases} 0.5x'' + 6x' + 50x = 0, \quad t > 0, \\ x(0) = 0.5 \\ x'(0) = -15. \end{cases}$$

The solution is $x(t) = \mathrm{e}^{-6t}(0.5\cos 8t - 1.5\sin 8t)$.

Problem 2.2.4 *a)* $u(t) = c_1\cos 2t + c_2\sin 2t + A\cos\omega t + B\sin\omega t$, with $A = \frac{1}{8-2\omega^2}$, $B = 0$, $c_1 = -\frac{1}{8-2\omega^2}$, $c_2 = 0$, that is, $u(t) = \frac{1}{2(\omega^2-4)}(\cos 2t - \cos\omega t)$; *b)* using L'Hôpital rule, $u(t) = \lim_{\omega\to 2}\frac{1}{2(\omega^2-4)}(\cos 2t - \cos\omega t) = \frac{1}{8}t\sin 2t$; *c)* $u(t) = c_1\cos 2t + c_2\sin 2t + Ct\cos 2t + Dt\sin 2t$, with $C = 0$, $D = \frac{1}{8}$, $c_1 = c_2 = 0$, that is, $u(t) = \frac{1}{8}t\sin 2t$.

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