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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 3

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3 Laplace transform

3.1 Properties of the Laplace transform

Problem 3.1.1 *a)* $\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}dt = 1$; integrating by parts, $\Gamma(2) = \int_0^\infty t\mathrm{e}^{-t}dt = \int_0^\infty \mathrm{e}^{-t}dt = 1$; with the change $t = z^2$, $\Gamma(1/2) = \int_0^\infty t^{-1/2}\mathrm{e}^{-t}dt = 2\int_0^\infty \mathrm{e}^{-z^2}dz = \sqrt{\pi}$. *b)* integrating by parts, $\Gamma(x+1) = \int_0^\infty t^x\mathrm{e}^{-t}dt = x\int_0^\infty t^{x-1}\mathrm{e}^{-t}dt = x\Gamma(x)$. *c)* $\lim_{x\to 0^+}\Gamma(x) = \lim_{x\to 0^+}\frac{\Gamma(x+1)}{x} = +\infty$. *d)* by induction using $\Gamma(n+1) = n\Gamma(n)$. *e)* also by induction: $\Gamma\left(n+\frac{3}{2}\right) = \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!(2n+1)}{2^{2n}n!2}\sqrt{\pi} = \frac{(2n+2)!}{2^{2n+2}(n+1)!}\sqrt{\pi}$.

Problem 3.1.2 *a)* $\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$; *b.i)* $\frac{1}{2}\sqrt{\pi}(s-3)^{-3/2}$; *b.ii)* $\sqrt{\frac{\pi}{s-1}}$.

Problem 3.1.3 *a)* $L(\mathrm{e}^{-at}f(t))(s) = \int_0^\infty \mathrm{e}^{-at}f(t)\mathrm{e}^{-st}dt = \int_0^\infty f(t)\mathrm{e}^{-(a+s)t}dt = L(f)(s+a)$.
b) $L(f(at))(s) = \int_0^\infty f(at)\mathrm{e}^{-st}dt = \frac{1}{a}\int_0^\infty f(z)\mathrm{e}^{-sz/a}dz = \frac{1}{a}L(f(t))\left(\frac{s}{a}\right)$.

Problem 3.1.4 *i)* $\frac{2}{s}+\frac{1}{s^2}-\frac{8}{s^3}+\frac{18}{s^4}$, $s>0$; *ii)* $\frac{1}{(s-a)^2}$, $s>a$; *iii)* $\frac{3\sqrt{\pi}}{4(s-1)^{5/2}}$; *iv)* $\frac{\Gamma(b+1)}{(s-a)^{b+1}}$, $s>a$; *v)* $\frac{3}{s-2}+\frac{2}{(s+1)^3}-\frac{12}{(s-1)^4}$, $s>2$; *vi)* $\frac{a}{s^2+a^2}$, $s>0$; *vii)* $\frac{s}{s^2+a^2}$, $s>0$; *viii)* $\frac{s+a}{b^2+(s+a)^2}$, $s>-a$; *ix)* $\frac{b}{b^2+(s+a)^2}$, $s>-a$; *x)* $\frac{2}{s(s^2+4)}$, $s>0$; *xi)* $\frac{s^2+2}{s(s^2+4)}$, $s>0$; *xii)* $\frac{6}{(s^2+1)(s^2+9)}$, $s>0$; *xiii)* $\frac{s^3+7s}{(s^2+1)(s^2+9)}$, $s>0$.

Problem 3.1.5 *a)* $\frac{d}{ds}[L(f)(s)] = \int_0^\infty f(t)\frac{d}{ds}e^{-st}\,dt = \int_0^\infty (-t)f(t)e^{-st}\,dt = L(-t\,f(t))(s)$;
b) by induction, $\frac{d^{n+1}}{ds^{n+1}}[L(f)(s)] = \frac{d}{ds}L((-t)^n f(t))(s) = L((-t)^{n+1}f(t))(s)$.

Problem 3.1.6 *i)* $-\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$, $s>0$; *ii)* $\frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{6as^2-2a^3}{(s^2+a^2)^3}$, $s>0$.

Problem 3.1.7 *a)* $L(f(t-a)H(t-a)) = \int_a^\infty f(t-a)\mathrm{e}^{-st}\,dt = \int_0^\infty f(z)\mathrm{e}^{-sz-sa}\,dz = \mathrm{e}^{-as}L(f(t))(s)$;
b) using that $t^3 = (t-3)^3+6(t-3)+9$, we obtain $\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)\mathrm{e}^{-3s}$.

Problem 3.1.8 *i)* $L^{-1}\left(\frac{1/2}{s-1}-\frac{1/2}{s+1}\right) = \frac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x}) = \operatorname{senh} x$; *ii)* $x\,\mathrm{e}^{-x}$;
iii) $L^{-1}\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right) = 1-(x+1)\,\mathrm{e}^{-x}$; *iv)* $\frac{x^{n-1}}{(n-1)!}$;
v) $L^{-1}\left(\frac{-1/2}{s-1}+\frac{1/2}{(s-1)^2}+\frac{s/2}{s^2+1}\right) = \frac{1}{2}((x-1)\mathrm{e}^x+\cos x)$; *vi)* $L^{-1}\left(\frac{4}{s+4}-\frac{4}{(s+4)^2}\right) =$

$4(1-x)e^{-4x}$; *vii)* $\cos(a(x-\pi/2))H(x-\pi/2)$, that is, $\cos(a(x-\pi/2))$ if $x \geq \pi/2$, 0 if $x < \pi/2$; *viii)* $\frac{1}{\sqrt{\pi x}}$; *ix)* $L^{-1}\left(\frac{-1}{s^2}+\frac{1/2}{s-1}-\frac{1/2}{s+1}\right) = -x+\sinh x$;
x) $L^{-1}\left(\frac{s+1}{(s+1)^2+9}+\frac{2}{(s+1)^2+9}+e^{-2s}(\frac{1}{s+2}-\frac{1}{s+3})\right) = (\cos 3x+\frac{2}{3}\sin 3x)e^{-x}+(e^{-2(x-2)}-e^{-3(x-2)})H(x-2)$.

Problem 3.1.9 Since $g'(x) = f(x)$ and $g(0) = 0$, we have $L(f)(s) = sL(g)(s)$.

Problem 3.1.10 $-\frac{d}{ds}\left(\frac{1}{s}L(\sin bt)(s+a)\right) = -\frac{d}{ds}\left(\frac{b}{s((s+a)^2+b^2)}\right) = \frac{b(3s^2+4as+a^2+b^2)}{s^2((s+a)^2+b^2)^2}$.

Problem 3.1.11 *a)* $L(f)(s) = \int_0^P e^{-st}f(t)\,dt+\int_P^\infty e^{-st}f(t)\,dt = \int_0^P e^{-st}f(t)\,dt+e^{-sP}\int_0^\infty e^{-sz}f(z)\,dz = \int_0^P e^{-st}f(t)\,dt+e^{-sP}L(f)(s)$; *b.i)* $\frac{1}{1-e^{-s}}\int_0^1 te^{-st}\,dt = \frac{e^s-s-1}{s^2(e^s-1)}$;
b.ii) $\frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\,dt-\int_1^2 e^{-st}\,dt\right) = \frac{e^s-1}{s(e^s+1)}$.

3.2 Linear equations and systems.

Problem 3.2.1 *i)* $sY(s)-1-3Y(s) = \frac{1}{s-2} \rightsquigarrow Y(s) = \frac{s-1}{(s-2)(s-3)} = -\frac{1}{s-1}+\frac{2}{s-2} \rightsquigarrow y(t) = 2e^{3t}-e^{2t}$; *ii)* $Y(s) = \frac{2}{(s+3)(s^2+4)} \rightsquigarrow y(t) = \frac{1}{13}(2e^{-3t}-2\cos 2t+3\sin 2t)$;
iii) $Y(s) = \frac{s^2+2s+9}{2(s-5)(s^2+9)} \rightsquigarrow y(t) = \frac{1}{34}(22e^{5t}-5\cos 3t+3\sin 3t)$;
iv) $Y(s) = \frac{s-3}{2(s-5)^2} \rightsquigarrow y(t) = (t+\frac{1}{2})e^{5t}$;
v) $Y(s) = \frac{s+2}{(s+1)^3} \rightsquigarrow y(t) = (t+\frac{t^2}{2})e^{-t}$;
vi) $Y(s) = \frac{1}{(s-2)(s+1)} \rightsquigarrow y(t) = \frac{1}{3}(e^{2t}-e^{-t})$;
vii) $Y(s) = \frac{s^2+s+1}{(s^2+1)(s^2+16)} \rightsquigarrow y(t) = \frac{1}{15}(\cos t-4\cos 4t)+\frac{1}{4}\sin(4t)$;
viii) $Y(s) = \frac{s^2+5s+7}{(s+1)^2(s+3)} \rightsquigarrow y(t) = \frac{1}{4}(e^{-3t}+3e^{-t}+6te^{-t})$;
ix) $Y(s) = \frac{2(s-3)^4+2}{(s-3)^5} \rightsquigarrow y(t) = \frac{1}{12}(24+t^4)e^{3t}$;
x) $Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} \rightsquigarrow y(t) = \frac{1}{6}(1+2e^{-t}-3e^{-2t}\cos(\sqrt{2}\,t)-2\sqrt{2}\,e^{-2t}\sin(\sqrt{2}\,t))$;
xi) $Y(s) = \frac{1}{s(s^2-2s+2)} \rightsquigarrow y(t) = \frac{1}{2}[1-e^t(\cos t-\sin t)]$;
xii) $Y(s) = \frac{1}{s^2-2s+2} \rightsquigarrow y(t) = e^t\sin t$.

Problem 3.2.2 *i)* $y(t) = e^{-2t}(c_1\cos 2t+c_2\sin 2t)+Ae^{-t} \rightsquigarrow y(t) = \frac{1}{10}[2e^{-t}-e^{-2t}(2\cos 2t+\sin 2t)]$, or also $Y(s) = \frac{1}{(s+1)(s^2+4s+8)} \rightsquigarrow y(t) = \cdots$;

ii) $y(t) = e^x(c_1 + c_2x) + A + (Bx + C)e^{2x} \rightsquigarrow y(x) = 1 + e^x(2x + 1) + e^{2x}(x - 2)$, or also $Y(s) = \dfrac{s^2 - 3s + 4}{s(s-2)^2(s-1)^2} \rightsquigarrow y(t) = \cdots$.

Problem 3.2.3 *i)* $Y(s) = \dfrac{s^2 - 4s + 3}{s(s^3 - 4s^2 - 5s)} \rightsquigarrow y(t) = \dfrac{4}{75}e^{5t} - \dfrac{4}{3}e^{-t} + \dfrac{96}{75} - \dfrac{3t}{5}$;
ii) $X(s) = \dfrac{1}{s^3 + s^2 - 6s} \rightsquigarrow x(t) = -\dfrac{1}{10}e^{2t} + \dfrac{1}{15}e^{-3t} - \dfrac{1}{6}$.

Problem 3.2.4
SOLUTION: $Y(s) = \dfrac{s^2 + s + 16}{(s^2 + 16)^2} \rightsquigarrow y(t) = \dfrac{1}{8}[(2 + t)\sin 4t]$.

Problem 3.2.5 $X(s) = \dfrac{k\omega}{(s^2 + \omega)(s^2 + \omega_0)} \rightsquigarrow x(t) = \dfrac{k}{\omega_0} \cdot \dfrac{\omega_0 \sin \omega t - \omega \sin \omega_0 t}{\omega_0^2 - \omega^2}$ if $\omega \neq \omega_0$;
$x(t) = \dfrac{k}{2\omega_0^2}(\sin \omega_0 t - \omega_0 t \cos \omega_0 t)$ if $\omega = \omega_0$.

Problem 3.2.6 $F(s) = \dfrac{2 - s + 2s^2 - s^3}{s^2(s^2 + 1)} \rightsquigarrow f(x) = 2x - 1$. Or also we can derive, $\begin{cases} f'' + f = 2x - 1 \\ f(0) = -1,\ f'(0) = 2, \end{cases}$ $f(x) = c_1 \sin x + c_2 \cos x + Ax + B$, and we obtain $c_1 = c_2 = 0$, $A = 2$, $B = -1$.

Problem 3.2.7 *i)* $\begin{cases} sX = 4X - Y \\ sY - 1 = 2X + Y \end{cases} \rightsquigarrow X(s) = -\dfrac{1}{s^2 - 5s + 6} \rightsquigarrow x(t) = -e^{3t} + e^{2t} \rightsquigarrow$ $y(t) = -e^{3t} + 2e^{2t}$;
ii) $x(t) = 4te^{3t},\ y(t) = (1 - 2t)e^{3t}$;
iii) $x(t) = e^{-t}(\cos \sqrt{2}t - \sqrt{2}\sin \sqrt{2}t),\ y(t) = e^{-t}(2\cos \sqrt{2}t + \sqrt{2}\sin \sqrt{2}t)$;
iv) $x(t) = \dfrac{2}{3}\sin 3t,\ y(t) = 1 + \dfrac{1}{3}(\sin 3t - 3\cos 3t)$;
v) $x(t) = y(t) = 1 - t$.
vi) $x(t) = e^{2t}\sinh t$, $y(t) = e^{2t}\cosh t$.
vii) $x(t) = \dfrac{1}{3}e^{5t} + \dfrac{2}{3}e^{-t}$, $y(t) = \dfrac{2}{3}e^{5t} - \dfrac{2}{3}e^{-t}$.
viii) $x(t) = e^{-3t}(2\cos 2t - \sin 2t)$, $y(t) = e^{-3t}(-\cos 2t - 2\sin 2t)$.
ix) $x(t) = \dfrac{1}{15}\left(9e^{2t} - 5e^{3t} - 4e^{-3t}\right)$, $y(t) = \dfrac{2}{15}\left(3e^{2t} - 5e^{3t} + 2e^{-3t}\right)$.

Problem 3.2.8 *i)* $\begin{cases} x'' - 2x' + x = 0 \\ x(0) = 0,\ x'(0) = -8 \end{cases} \rightsquigarrow x(t) = -8te^t \rightsquigarrow y(t) = 2e^t - 4te^t$;
ii) $\begin{cases} x'' - 7x' + 12x = -12e^t \\ x(0) = -1,\ x'(0) = 2 \end{cases} \rightsquigarrow x(t) = -2e^t + e^{4t} \rightsquigarrow y(t) = \dfrac{2}{3}(e^{4t} - e^t)$.

Problem 3.2.9 $F(s) = \dfrac{4}{s^2} + \dfrac{F(s)}{s^2 + 1} \rightsquigarrow F(s) = \dfrac{4(s^2 + 1)}{s^4} \rightsquigarrow f(t) = 4t + \dfrac{2}{3}t^3$.

Problem 3.2.10 $sF(s) = -1 = \dfrac{1}{s} - \dfrac{F(s)}{s + 2} \rightsquigarrow F(s) = \dfrac{s + 2}{s(s + 1)} \rightsquigarrow f(x) = 2 - e^{-x}$.

Problem 3.2.11 If we derive twice we obtain a second order ODE: $1 - \dfrac{1}{4}f'(x) = \displaystyle\int_0^x f(u)\,du \rightsquigarrow$ $-\dfrac{1}{4}f'' = f \Rightarrow f'' + 4f = 0$. Two initial data are obtained by substituting into the original equation and in the first derivative, $f(0) = 0$, $f'(0) = 4$. The solution is then $f(x) = 2\sin 2x$. Alternatively, transforming by Laplace we arrive to an algebraic equation, using convolution, $\frac{1}{s^2} - \frac{1}{4}F(s) = \frac{F(s)}{s^2} \rightsquigarrow F(s) = \frac{4}{s^2+4}$.

Problem 3.2.12 *i)* The independent term is $f(t) = t + 1 - 2(t-1)H(t-1)$; the transformed equation is $(s^2+s)Y = -1 - s + \frac{1}{s} + \frac{1}{s^2} - \frac{2}{s^2}e^{-s}$; and the solution is $y(t) = \frac{t^2}{2} - 1 - (t^2 - 4t + 5 - 2e^{1-t})H(t-1)$. *ii)* The independent term is $f(t) = \cos 2t - \cos(2(t-2\pi))H(t-2\pi)$; the transformed equation is $(s^2+4)Y = \frac{s}{s^2+4}(1 - e^{-2\pi s})$; and the solution finally is $y(t) = \frac{1}{4}\sin 2t\Big(t - (t-2\pi)H(t-2\pi)\Big)$.

Problem 3.2.13 The independent term is $f(t) = e^{\pi/2}e^{t-\pi/2}H(t-\pi/2) - e^{\pi}e^{t-\pi}H(t-\pi)$; the transformed equation is $(s^2-1)X = \frac{1}{s-1}(e^{\pi/2}e^{-\pi s/2} - e^{\pi}e^{-\pi s})$; and the solution is $x(t) = \frac{1}{4}(e^{\pi-t} - e^t + 2(t-\pi/2)e^t)H(t-\pi/2) - \frac{1}{4}(e^{2\pi-t} - e^t + 2(t-\pi)e^t)H(t-\pi)$.

Problem 3.2.14 *i)* $s^2Y(s) = e^{-as} \rightsquigarrow y(t) = (t-a)H(t-a)$. *ii)* $Y(s) = \frac{e^{-s}+e^{-2s}}{s+8} \rightsquigarrow y(t) = e^{-8(t-1)}H(t-1) + e^{-8(t-2)}H(t-2)$. *iii)* $Y(s) = \frac{1+4e^{-3\pi s/2}}{s^2+1} \rightsquigarrow y(t) = \sin t + 4\sin(t-3\pi/2)H(t-3\pi/2) = \sin t + 4\cos t H(t-3\pi/2)$. *iv)* $Y(s) = \frac{1}{s} + \frac{e^{-2\pi s}}{s^2+1} \rightsquigarrow y(t) = 1 + \sin(t-2\pi)H(t-2\pi) = 1 + \sin t H(t-2\pi)$.

Problem 3.2.15 *i)* $Y(s) = \frac{E_0}{Ls+R} \rightsquigarrow I(t) = \frac{E_0}{L}e^{-Rt/L}$; *ii)* $Y(s) = \frac{E_0}{(Ls+R)(s^2+\omega^2)} \rightsquigarrow I(t) = \frac{E_0}{R^2+L^2\omega^2}(-\omega L\cos\omega t + R\sin\omega t) + \frac{E_0 L\omega}{R^2+L^2\omega^2}e^{-Rt/L}$.

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