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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 4

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4 Method of separation of variables

4.1 Separation of variables

Problem 4.1.1 *i)* Writing $u(r,t) = R(r)T(t)$ we obtain $(rR')' + \lambda rR = 0$, $T' + \lambda kT = 0$.
ii) $u(r,\theta) = R(r)H(\theta) \rightsquigarrow (rR')' - \dfrac{\lambda}{r}R = 0$, $H'' + \lambda H = 0$.
iii) $u(x,t) = X(x)T(t) \rightsquigarrow kX'' - v_0 X' + \lambda X = 0$, $T' + \lambda kT = 0$.
iv) $u(x,y) = X(x)Y(y) \rightsquigarrow X'' + \lambda X = 0$, $Y'' - \lambda Y = 0$.
v) $u(r,t) = R(r)T(t) \rightsquigarrow (r^2R')' + \lambda r^2 R = 0$, $T' + \lambda kT = 0$.
vi) $u(x,t) = X(x)T(t) \rightsquigarrow X^{iv)} + \lambda X = 0$, $T' + \lambda kT = 0$.
vii) $u(x,t) = X(x)T(t) \rightsquigarrow X'' + \lambda X = 0$, $T'' + \lambda c^2 T = 0$.

Problem 4.1.2

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin nx \sinh n(\pi - y);$$
$$2\sin 3x = u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx \sinh n\pi \;\Rightarrow\; a_3 = \frac{2}{\sinh 3\pi}, \quad a_n = 0\ \forall n \neq 3 \;\Rightarrow$$
$$u(x,y) = \frac{2}{\sinh 3\pi} \sin 3x \sinh 3(\pi - y).$$

Problem 4.1.3

$$\varphi(x,y) = b_0 y + \sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{L}) \sinh(\frac{n\pi y}{L});$$
$$f(x) = \varphi(x,L) = b_0 L + \sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{L}) \sinh(n\pi) \;\Rightarrow$$
$$b_0 = \frac{1}{L^2}\int_0^L f(s)\,ds, \qquad b_n = \frac{2}{L\sinh(n\pi)}\int_0^L f(s)\cos(\frac{n\pi s}{L})\,ds \quad n \geq 1.$$

Problem 4.1.4

a) The compatibility condition is $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial \nu} = 0$, that is $\displaystyle\int_0^L f(x)\,dx = 0$.

b)

$$u(x,y) = b_0 + \sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{L}) \cosh(\frac{n\pi y}{L});$$
$$f(x) = \frac{\partial u}{\partial y}(x,H) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos(\frac{n\pi x}{L}) \sinh(\frac{n\pi H}{L}) \;\Rightarrow$$
$$b_0 \in \mathbb{R}, \quad \int_0^L f(s)\,ds = 0, \quad b_n = \frac{2L}{Ln\pi \sinh(n\pi H/L)}\int_0^L f(s)\cos(\frac{n\pi s}{L})\,ds \quad n \geq 1.$$

c) If $u(x,y) = \lim_{t\to\infty} v(x,y,t)$, where v is the solution of the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v, & 0 < x < L,\ 0 < y < H,\ t > 0, \\ \dfrac{\partial v}{\partial x}(0,y,t) = \dfrac{\partial v}{\partial y}(x,0,t) = \dfrac{\partial v}{\partial x}(L,y,t) = 0, & \\ \dfrac{\partial v}{\partial y}(x,H,t) = f(x), & \\ v(x,y,0) = g(x,y), & \end{cases}$$

then, by the conservation of energy,

$$\int_0^L \int_0^H g(x,y)\, dydx = \int_0^L \int_0^H v(x,y,0)\, dydx = b_0 LH.$$

That is, b_0 is equal to the medium value of the initial data g on the rectangle.

Observe that the conservation of energy is deduced from the equation and the boundary conditions:

$$E(t) = \int_\Omega v \;\Rightarrow\; E'(t) = \int_\Omega \Delta v = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0 + 0 + 0 + \int_0^L f(x)\, dx = 0.$$

Problem 4.1.5

a) The stationary or equilibrium solution satisfies the problem $\begin{cases} kv'' - \alpha v = 0, & 0 < x < L, \\ v(0) = v(L) = 0. & \end{cases}$

The only solution is $v(x) = 0$.

b) We separate variables and obtain

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L) \mathrm{e}^{-(\alpha + kn^2\pi^2/L^2)t}.$$

The initial condition implies $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(s)\sin(n\pi s/L)\, ds$. It is clear that, independently of the initial data, we obtain $\lim_{t\to\infty} u(x,t) = 0$, since all the exponents are negative.

Problem 4.1.6

Problem 4.1.7

a) We separate variables and obtain

$$u(x,t) = \sum_{n=1}^{\infty} \Big(a_n \sin(\omega_n t) + b_n \cos(\omega_n t)\Big) \sin(\sqrt{\lambda_n}\, x),$$

where $\lambda_n = n^2\pi^2/L^2$ y $\omega_n = \sqrt{T_0\lambda_n/\rho}$, while a_n and b_n are the Fourier sine coefficients of $g(x)/\omega_n$ and $f(x)$, respectively. We write the time dependent part as $\alpha_n \cos[\omega_n(t - \delta_n)]$, and develop the cosine of the sum, then we obtain

$$\begin{cases} \alpha_n \sin(\omega_n \delta_n) = a_n \\ \alpha_n \cos(\omega_n \delta_n) = b_n \end{cases} \quad\Rightarrow\quad \begin{cases} \alpha_n = \sqrt{a_n^2 + b_n^2} \\ \delta_n = \dfrac{1}{\omega_n} \operatorname{arctg}(\dfrac{a_n}{b_n}) \end{cases}$$

b) For every harmonic, (that is, for fixed n), in the antinodes we have $\sin(\sqrt{\lambda_n}\, x) = \pm 1$, which implies: $x = \dfrac{(2j+1)L}{2n}$, $j = 0, 1, \cdots, n-1$.

c)

$$\Big(\frac{\partial U_n}{\partial t}\Big)^2 = \alpha_n^2 \omega_n^2 \sin^2(\omega_n(t-\delta_n)) \sin^2(\sqrt{\lambda_n}\, x),$$

$$\Big(\frac{\partial U_n}{\partial x}\Big)^2 = \alpha_n^2 \lambda_n \cos^2(\omega_n(t-\delta_n)) \cos^2(\sqrt{\lambda_n}\, x),$$

$$\int_0^L \sin^2(\sqrt{\lambda_n}\, x)\, dx = \int_0^L \cos^2(\sqrt{\lambda_n}\, x)\, dx = \frac{L}{2},$$

$$E(t) = \frac{1}{2}\rho \alpha_n^2 \omega_n^2 \frac{L}{2} \sin^2(\omega_n(t-\delta_n)) + \frac{1}{2} T_0 \alpha_n^2 \lambda_n \frac{L}{2} \cos^2(\omega_n(t-\delta_n)) = \frac{1}{4L} T_0 n^2 \pi^2 \alpha_n^2.$$

d) $a_1 = 0 \quad \rightsquigarrow \quad \alpha_1 = b_1 = \dfrac{2}{L}\displaystyle\int_0^L f(s) \sin(n\pi s/L)\, ds = \dfrac{8A}{\pi^2} \quad \rightsquigarrow \quad E_1 = \dfrac{16 T_0 A^2}{L\pi^2}.$

Problem 4.1.8

a) Deriving and integrating by parts the second term,

$$\begin{aligned} E'(t) &= \rho \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx + T_0 \int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx = \rho \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx - T_0 \int_0^L \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\, dx \\ &= \int_0^L \frac{\partial u}{\partial t}\Big(\rho \frac{\partial^2 u}{\partial t^2} - T_0 \frac{\partial^2 u}{\partial x^2}\Big)\, dx = 0, \end{aligned}$$

since $u(0,t) = u(L,t) = 0$ implies $\left.\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial t}\right|_0^L = 0$.

b) Using linearity, it is enough to prove that the unique solution with zero initial data is the zero solution. Since $E(t) = \textit{constant}$, then $E(t) = E(0)$ for every $t \geq 0$. But if $f = g = 0$ in the previous problem, the initial energy is

$$E(0) = \frac{\rho}{2}\int_0^L (g(x))^2\, dx + \frac{T_0}{2}\int_0^L (f'(x))^2\, dx = 0.$$

So, for every $t \geq 0$,

$$\frac{\rho}{2}\int_0^L \Big(\frac{\partial u}{\partial t}\Big)^2 dx + \frac{T_0}{2}\int_0^L \Big(\frac{\partial u}{\partial x}\Big)^2 dx = 0 \;\Rightarrow\; \frac{\partial u}{\partial x} = 0 \;\Rightarrow\; u = \textit{constant} \;\Rightarrow\; u = 0.$$

Problem 4.1.9 Separating variables

$$u(x,t) = a_0 t + b_0 + \sum_{n=1}^{\infty} \Big(a_n \sin(cnt) + b_n \cos(cnt)\Big) \cos(nx).$$

The initial position implies $b_n = 0$ for every $n \geq 0$. The initial velocity is then,

$$8\sin^2 x = 4(1-\cos(2x)) = \frac{\partial u}{\partial t}(x,0) = a_0 + \sum_{n=1}^{\infty} a_n cn \cos(nx),$$

so $a_0 = 4$, $a_2 = -2/c$, $a_n = 0$ for every $n \neq 0, 2$. Then, the solution is

$$u(x,t) = 4t - \frac{2}{c}\sin(2ct)\cos(2x).$$

Problem 4.1.10 We separate variables $u(x,t) = X(x)T(t)$, the time-dependent equation is $T'' + \alpha T' + \lambda_n c^2 T = 0$, where the eigenvalues are obtained in the spatial problem, $\lambda_n = n^2$, $n \geq 1$. The associated characteristic equation in T has the solutions $r = \dfrac{-\alpha \pm \sqrt{\alpha^2 - 4c^2n^2}}{2}$. So it is important the sign of $\alpha^2 - 4c^2n^2$. Since the initial condition is $u(x,0) = \sin(2x)$, we only need the value for $n = 2$. The solution is then:

$$\begin{aligned}
&i) \text{ si } \alpha < 4c \rightsquigarrow && u(x,t) = (a_2 \sin(zt) + b_2 \cos(zt))\mathrm{e}^{-\alpha t/2} \sin 2x, \\
&ii) \text{ si } \alpha = 4c \rightsquigarrow && u(x,t) = (c_2 t + g_2)\mathrm{e}^{-\alpha t/2} \sin 2x, \\
&iii) \text{ si } \alpha > 4c \rightsquigarrow && u(x,t) = (h_2 \mathrm{e}^{wt} + j_2 \mathrm{e}^{-wt})\mathrm{e}^{-\alpha t/2} \sin 2x,
\end{aligned}$$

where $z = \dfrac{\sqrt{16c^2 - \alpha^2}}{2}$, $w = \dfrac{\sqrt{\alpha^2 - 16c^2}}{2}$. The initial conditions imply:

$$a_2 = \frac{\alpha}{2z}, \quad b_2 = 1, \quad c_2 = \frac{\alpha}{2}, \quad g_2 = 1, \quad h_2 = \frac{2w + \alpha}{4w}, \quad j_2 = \frac{2w - \alpha}{4w}.$$

Problem 4.1.11

a) Let us separate $u(r,\theta) = R(r)H(\theta)$, the angular part gives us the eigenvalues $\lambda_n = n^2$, $n \geq 0$. With them we solve the radial part, an equidimensional equation: $r^2 R'' + rR' - \lambda_n R = 0$. We use the non singularity condition at $r = 0$ and obtain: $R(r) = \begin{cases} r^n, & n \geq 1, \\ 1, & n = 0. \end{cases}$ With the boundary condition, the solution is:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi)[\sin n\phi \sin n\theta + \cos n\phi \cos n\theta]\, d\phi \\
&= \int_{-\pi}^{\pi} f(\phi) P(r,\theta,\phi)\, d\phi,
\end{aligned}$$

where Poisson's Integral Formula is:

$$\begin{aligned}
P(r,\theta,\phi) &= \frac{1}{2\pi} \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [\sin n\phi \sin n\theta + \cos n\phi \cos n\theta]\right] \\
&= \frac{1}{2\pi} \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi)\right].
\end{aligned}$$

b) We define $z = \dfrac{r}{a}\mathrm{e}^{(\theta-\phi)i}$, and obtain

$$\begin{aligned}
P(r,\theta,\phi) &= \frac{1}{2\pi} \mathcal{R}e \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \mathrm{e}^{n(\theta-\phi)i}\right] = \frac{1}{2\pi} \mathcal{R}e \left[1 + 2\sum_{n=1}^{\infty} z^n\right] \\
&= \frac{1}{2\pi} \mathcal{R}e \left[1 + \frac{2z}{1-z}\right] = \frac{1}{2\pi} \mathcal{R}e \left[\frac{1+z}{1-z}\right] = \frac{1}{2\pi} \frac{1 - |z|^2}{1 - 2\mathcal{R}e\, z + |z|^2} \\
&= \frac{1}{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2}.
\end{aligned}$$

Then

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} f(\phi)\, d\phi.$$

c) At $r = 0$, we have $u(0,\theta) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\phi)\,d\phi$.

Problem 4.1.12 Since $\sin^3\theta = \dfrac{1}{4}(3\sin\theta - \sin 3\theta)$, it is easy to obtain the solution $u(r,\theta) = \dfrac{1}{4}(3r\sin\theta - r^3\sin 3\theta)$.

Problem 4.1.13

a) We use odd symmetry around the diameter: $\theta = 0$, and obtain the solution from the solution on the entire disk using the boundary condition obtained by odd reflection: $\widetilde{g}(\theta) = \begin{cases} g(\theta), & 0 < \theta < \pi, \\ -g(-\theta), & -\pi < \theta < 0. \end{cases}$ Then

$$\begin{aligned} u(r,\theta) &= \int_{-\pi}^{\pi} \widetilde{g}(\phi)P(r,\theta,\phi)\,d\phi \\ &= \int_{0}^{\pi} g(\phi)P(r,\theta,\phi)\,d\phi - \int_{-\pi}^{0} g(-\phi)P(r,\theta,\phi)\,d\phi \\ &= \int_{0}^{\pi} g(\phi)[P(r,\theta,\phi) - P(r,\theta,-\phi)]\,d\phi. \end{aligned}$$

Also, we can separate variables directly and obtain:

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{2}{\pi}\int_0^{\pi} g(\phi)\sin n\phi\,d\phi\,\sin n\theta.$$

b) Now, we use even symmetry, the solution is

$$u(r,\theta) = \int_0^{\pi} g(\phi)[P(r,\theta,\phi) + P(r,\theta,-\phi)]\,d\phi.$$

Or also,

$$u(r,\theta) = \frac{1}{\pi}\int_0^{\pi} g(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{2}{\pi}\int_0^{\pi} g(\phi)\cos n\phi\,d\phi\,\cos n\theta.$$

Problem 4.1.14 We separate variables and obtain the eigenvalues $\lambda_n = 9n^2$, $n \geq 1$, and the solution

$$u(r,\theta) = \sum_{n=1}^{\infty} \alpha_n r^{3n}\sin(3n\theta).$$

The boundary condition implies:

$$\alpha_n = \frac{6}{\pi a^{3n}}\int_0^{\pi/3} f(s)\sin(3ns)\,ds.$$

Problem 4.1.15 *i*) The radial equation (equidimensional), $r^2R'' + rR' - n^2R = 0$, has the following solutions that satisfy the condition $R(b) = 0$,

$$R(r) = \begin{cases} (b/r)^n - (r/b)^n, & n \geq 1, \\ \log(b/r), & n = 0. \end{cases}$$

So, the solution has the form

$$u(r,\theta) = \beta_0 \log(b/r) + \sum_{n=1}^{\infty} [\alpha_n \sin n\theta + \beta_n \cos n\theta][(b/r)^n - (r/b)^n].$$

The condition at $r = a$ implies

$$\alpha_n = \frac{1}{\pi[(b/a)^n - (a/b)^n]} \int_{-\pi}^{\pi} f(s) \sin ns\, ds, \quad n \geq 1,$$
$$\beta_0 = \frac{1}{2\pi \log(b/a)} \int_{-\pi}^{\pi} f(s)\, ds,$$
$$\beta_n = \frac{1}{\pi[(b/a)^n - (a/b)^n]} \int_{-\pi}^{\pi} f(s) \cos ns\, ds, \quad n \geq 1.$$

ii) We use linearity to write $u = u_1 + u_2$, where

$$\begin{cases} \Delta u_1 = 0, \\ u(a,\theta) = f(\theta), \\ u(b,\theta) = 0, \end{cases} \qquad \begin{cases} \Delta u_2 = 0, \\ u(a,\theta) = 0, \\ u(b,\theta) = g(\theta). \end{cases}$$

We have already solved u_1. The solution u_2 is obtained in an analogous way, interchanging a by b. Now we join both:

$$u(r,\theta) = \int_{-\pi}^{\pi} [f(s) Q_1(r,\theta,s) + g(s) Q_2(r,\theta,s)]\, ds,$$

where

$$Q_1(r,\theta,s) = \frac{1}{2\pi} \frac{\log(b/r)}{\log(b/a)} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(b/r)^n - (r/b)^n}{(b/a)^n - (a/b)^n} \cos(n(\theta - s)),$$
$$Q_2(r,\theta,s) = \frac{1}{2\pi} \frac{\log(r/a)}{\log(b/a)} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^n} \cos(n(\theta - s)).$$

iii)

$$u(r,\theta) = \int_{-\pi}^{\pi} g(s) Q_3(r,\theta,s)\, ds,$$

where

$$Q_3(r,\theta,s) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(r/a)^n + (a/r)^n}{(b/a)^n + (a/b)^n} \cos(n(\theta - s)).$$

iv) The compatibility condition is $\int_{-\pi}^{\pi} f(s)\, ds = 0$, that is, the total flux through the boundary is zero. The solution is

$$u(r,\theta) = \beta_0 + \int_{-\pi}^{\pi} f(s) Q_4(r,\theta,s)\, ds,$$

where

$$Q_4(r,\theta,s) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{a[(r/b)^n + (b/r)^n]}{n[(a/b)^n - (b/a)^n]} \cos(n(\theta - s)),$$

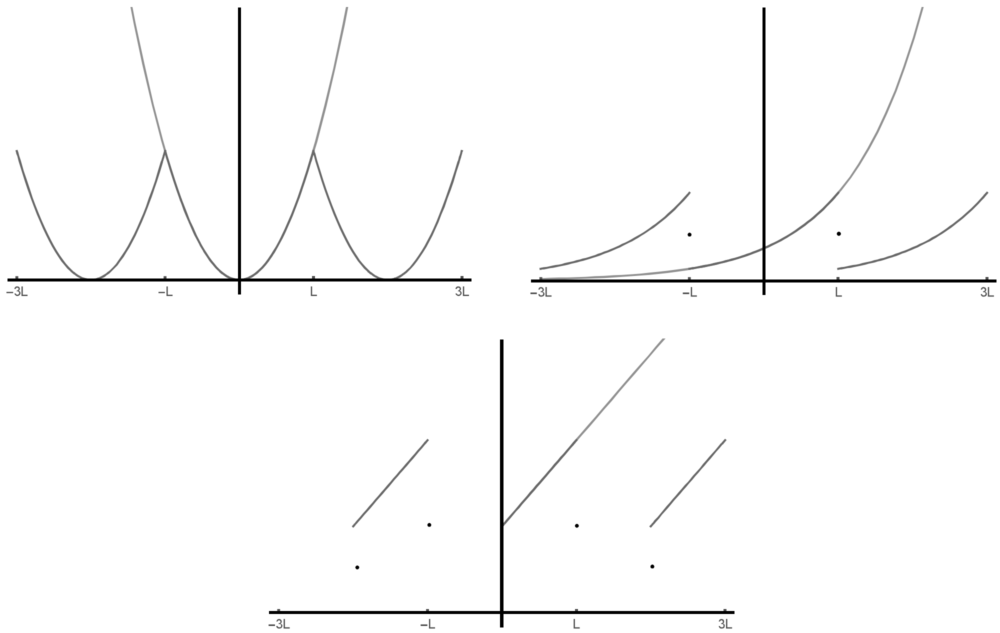
and $\beta_0 \in \mathbb{R}$ is a free parameter; we do not have uniqueness.

Problem 4.1.16 The solution is

$$u(r,\theta) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(r/a)^{2n} - (a/r)^{2n}}{(b/a)^{2n} - (a/b)^{2n}} \int_0^{\pi/2} f(s) \sin(2ns)\, ds\, \sin(2n\theta).$$

4.2 Fourier series

Problem 4.2.1



Problem 4.2.2 *i*) $S(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$. *ii*) $S(x) = \sum_{k=0}^{\infty} \frac{6}{(2k+1)\pi} \sin((2k+1)x) - \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(4kx)$.

iii) $S(x) = \sum_{k=0}^{\infty} \left(\frac{2\pi}{2k+1} - \frac{8}{\pi(2k+1)^3} \right) \sin((2k+1)x) - \sum_{k=1}^{\infty} \frac{\pi}{k} \sin(2kx)$.

iv) $S(x) = \sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2-1)} \sin(2kx)$.

Problem 4.2.3

a) Let us define $f(x) = -f(L-x)$. Since we have $\cos((n\pi(L-y)/L) = (-1)^n \cos(n\pi y/L)$, with the change of variables $L - x = y$ in the integral from $L/2$ and L, we have:

$$\begin{aligned} b_n = & \frac{2}{L} \int_0^L f(s) \cos(n\pi s/L)\, ds \\ = & \frac{2}{L} \int_0^{L/2} f(s) \cos(n\pi s/L)\, ds + \frac{2}{L} \int_0^{L/2} f(y)(-1)^n \cos(n\pi y/L)\, dy = 0, \end{aligned}$$

in the case where n is odd.

b) In fact, for even n, that is $n = 2k$, the corresponding coefficient is:

$$b_{2k} = \frac{4}{L} \int_0^{L/2} f(s) \cos(2k\pi s/L)\, ds.$$

Also, we can calculate directly the Fourier cosine coefficients on $[0, L/2]$:

$$c_m = \frac{4}{L} \int_0^{L/2} f(s) \cos(2m\pi s/L)\, ds,$$

and obtain the same series.

Problem 4.2.4 We cannot differentiate term by term the Fourier sine series of $f(x)$ and obtain the Fourier cosine series of $f'(x)$ because $f(x)$ is discontinuous at $x = 2k$, $k \in \mathbb{Z}$.

Problem 4.2.5 We consider the Fourier cosine series: $\mathrm{e}^x = \sum_{n=0}^{\infty} A_n \cos(n\pi x/L)$. We derive and obtain the Fourier sine series $\mathrm{e}^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin(n\pi x/L)$. Let us differentiate again, in the right way:

$$\mathrm{e}^x = \frac{1}{L}(\mathrm{e}^L - 1) + \sum_{n=1}^{\infty} \left[-\left(\frac{n\pi}{L}\right)^2 A_n + \frac{2}{L}((-1)^n \mathrm{e}^L - 1) \right] \cos(n\pi x/L).$$

Both Fourier cosine series must be equal, so we deduce that $A_n = \dfrac{2L((-1)^n \mathrm{e}^L - 1)}{L^2 + n^2\pi^2}$, $n \geq 1$, $A_0 = \dfrac{1}{L}(\mathrm{e}^L - 1)$.

Problem 4.2.6 Let us consider the Fourier sine series: $\cosh x = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$. Deriving we obtain the Fourier cosine series

$$\sinh x = \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh L - 1) \right] \cos(n\pi x/L).$$

Deriving again,

$$\cosh x = \sum_{n=1}^{\infty} \left(-\frac{n\pi}{L} \right) \left[\frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh L - 1) \right] \sin(n\pi x/L).$$

Now the two Fourier sine series are equal, so we deduce that $b_n = \dfrac{2n\pi(1 - (-1)^n \cosh L)}{L^2 + n^2\pi^2}$.

– *ERC*–