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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 5

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5 Sturm-Liouville problems

5.1 Eigenvalues and eigenfunctions

Problem 5.1.1 *i)* $\lambda_n = n^2\pi^2$, $n \geq 1$, $\varphi_n(x) = \sin(n\pi x)$. *ii)* $\lambda_n = n^2\pi^2$, $n \geq 0$, $\varphi_n(x) = \cos(n\pi x)$. *iii)* $\lambda_n = (n + 1/2)^2\pi^2$, $n \geq 0$, $\varphi_n(x) = \sin((n + 1/2)\pi x)$.

Problem 5.1.2 *i)* $\lambda_n = n^2\pi^2 - 3/4$, $n \geq 1$, $y_n(x) = e^{-x/2}\sin(n\pi x)$. *ii)* $\lambda_0 = -1$, $y_0(x) = 1$; $\lambda_n = (4n^2\pi^2 - 3)/12$, $n \geq 1$, $y_n(x) = e^{3x/2}(\sin(n\pi x) - \frac{2n\pi}{3}\cos(n\pi x))$. *iii)* $\lambda_0 = 0$, $y_0(x) = xe^{-x}$; $\lambda_n = -\mu_n^2$, $n \geq 1$, where $\mu_n = n$–th positive zero of the function $f(x) = \operatorname{tg} x - x$, $y_n(x) = e^{-x}\sin(\mu_n x)$.

Problem 5.1.3

$$\lambda = \frac{\displaystyle\int_0^1 (\phi')^2 + \int_0^1 x^2\phi^2}{\displaystyle\int_0^1 \phi^2} \geq 0$$

If $\lambda = 0$ we have $x^2\phi^2 = 0 \Rightarrow \phi = 0$. So $\lambda = 0$ is not an eigenvalue.

Problem 5.1.4 *i)* We use the test function $\phi(x) = 1 - x^2$, which satisfies $\phi'(0) = \phi(1) = 0$. Therefore

$$\lambda_1 \leq \frac{\displaystyle\int_0^1 (\phi')^2 + \int_0^1 x^2\phi^2}{\displaystyle\int_0^1 \phi^2} = \frac{37}{14} \approx 2.64$$

(With the test function $\phi(x) = \cos(\pi x/2)$ the bound is better, $\lambda_1 \leq \dfrac{\pi^2}{16} + \dfrac{1}{3} - \dfrac{2}{\pi^2} \approx 2.59$). *ii)* Con $\phi(x) = x^2 - 2$,

$$\lambda_1 \leq \frac{-\Big[\phi\phi'\Big]_0^1 + \displaystyle\int_0^1 (\phi')^2 + \int_0^1 x\phi^2}{\displaystyle\int_0^1 \phi^2} = \frac{135}{86}.$$

iii) With $\phi(x) = 2x^2 - 3x$,

$$\lambda_1 \leq \frac{-\Big[\phi\phi'\Big]_0^1 + \displaystyle\int_0^1 (\phi')^2}{\displaystyle\int_0^1 \phi^2} = \frac{25}{6}.$$

5.2 Generalized Fourier series

Problem 5.2.1

a) We write $u(x,t) = X(x)T(t)$ and obtain the following equation for x that is not a Sturm-Liouville equation: $kX'' - V_0X' + \lambda X = 0$.

b) The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{V_0x/2k}\sin(n\pi x/L)e^{-(\frac{V_0^2}{4k} + \frac{kn^2\pi^2}{L^2})t}.$$

The initial condition implies: $a_n = \dfrac{2}{L}\displaystyle\int_0^L (f(x)e^{-V_0x/2k})\sin(n\pi x/L)\,dx.$

Problem 5.2.2 We must have $H = p$, $\alpha H = p'$, that implies $H(x) = e^{\int^x \alpha(s)ds}$.

Problem 5.2.3

a) For $H(x) = 1/x$ we have $(x\phi')' + \dfrac{\lambda}{x}\phi = 0$.

b) For every pair (φ, λ) we have $\lambda = \dfrac{\displaystyle\int_0^b (\varphi')^2 x\, dx}{\displaystyle\int_1^b \varphi^2 \frac{1}{x}\, dx} \geq 0$. If $\lambda = 0$ then $\varphi' = 0$, and this leads to $\varphi = 0$.

c) $\lambda_n = n^2\pi^2/(\log b)^2$, $n \geq 1$, $\varphi_n(x) = \sin(n\pi \log x/\log b)$.

d) The eigenfunctions are orthogonal with respect to the weight $\sigma(x) = 1/x$.

$$\int_1^b \sin(\frac{n\pi \log x}{\log b})\, \sin(\frac{m\pi \log x}{\log b})\, \frac{1}{x}\, dx = \frac{\log b}{\pi} \int_0^\pi \sin(n\pi z)\, \sin(m\pi z)\, dz = 0, \qquad n \neq m.$$

e) It is clear that $\sin(n\pi \log x/\log b) = 0$ at the points $x = b^{m/n}$, $m = 1, 2, \cdots, n-1$. (Or also we can see that the zeros are $x = b^z$, where the z's are the zeros of $\sin(nz)$ on $0 < z < \pi$).

Problem 5.2.4

$$u(r,t) = \sum_{n=1}^{\infty} a_n J_0(\eta_{0,n} r/a) e^{-(k\eta_{0,n}^2/a^2)t},$$

where $\eta_{0,n} = n$–th zero of the Bessel function J_0, and the coefficients are obtained by

$$a_n = \frac{\displaystyle\int_0^a f(r) J_0(\eta_{0,n} r/a) r\, dr}{\displaystyle\int_0^a J_0^2(\eta_{0,n} r/a) r\, dr}.$$

Problem 5.2.5

a) The problem represents the vibration of a string tied up at the end points, with a reaction force αu, if $\alpha > 0$ (or damping if $\alpha < 0$), and friction βu_t (if $\beta < 0$) proportional to the velocity.

b) Writing $u(x,t) = X(x)T(t)$ we have

$$T_0 \frac{X''(x)}{\rho(x) X(x)} + \frac{\alpha(x)}{\rho(x)} = \frac{H''(t)}{H(t)} - \frac{\beta(x) H'(t)}{\rho(x) H(t)}$$

So, we need $\beta(x)/\rho(x) = \textit{constant}$.

c) If now $\beta = c\rho$, the separate equations are

$$T_0 X'' + (\alpha + \lambda\rho) X = 0, \qquad H'' - kH' + \lambda H = 0$$

The solutions of the time problem are, depending on the values of λ:

$$T(t) = \begin{cases} e^{kt/2}\left[c_1 \sinh(t\sqrt{k^2/4 - \lambda}) + c_2 \cosh(t\sqrt{k^2/4 - \lambda})\right] & \text{if} \quad \lambda < k^2/2, \\ e^{kt/2}(c_1 t + c_2) & \text{if} \quad \lambda = k^2/2, \\ e^{kt/2}\left[c_1 \sin(t\sqrt{\lambda - k^2/4}) + c_2 \cos(t\sqrt{\lambda - k^2/4})\right] & \text{if} \quad \lambda > k^2/2. \end{cases}$$

Problem 5.2.6 Solving the spatial equation (obtained after applying the separate variables method) we have that $\lambda_n = n^2\pi^2/L^2$, $n \geq 1$, $X_n(x) = \sin(n\pi x/L)$. On the other hand the temporal equation is: $T'' + aT' + (b+\lambda)T = 0$. The solution is different according to the values of λ in relation with a and b. So, if $\pi^2/L^2 > a^2/4 - b$, the solution of the telegraph problem is

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-at/2} \left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L),$$

where $w_n = \sqrt{\dfrac{n^2\pi^2}{L^2} - \dfrac{a^2}{4} + b}$, and the coefficients are

$$A_n = \frac{aB_n}{2w_n}, \quad B_n = \frac{2}{L}\int_0^L f(s)\sin(n\pi s/L)\,ds.$$

Moreover, if there exists $M \in \mathbb{N}$ such that $M - 1 < \dfrac{L}{\pi}\sqrt{\dfrac{a^2}{4} - b} < M$, the solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{M-1} \mathrm{e}^{-at/2} \left[C_n \sinh(z_n t) + D_n \cosh(z_n t)\right] \sin(n\pi x/L) \\ &+ \sum_{n=M}^{\infty} \mathrm{e}^{-at/2} \left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L), \end{aligned}$$

where $z_n = \sqrt{\dfrac{a^2}{4} - b - \dfrac{n^2\pi^2}{L^2}}$, and the new coefficients for n between 1 and $M-1$ are the following

$$C_n = \frac{aD_n}{2z_n}, \quad D_n = \frac{2}{L}\int_0^L f(s)\sin(n\pi s/L)\,ds.$$

Finally, if there exists $M \in \mathbb{N}$ with $\dfrac{L}{\pi}\sqrt{\dfrac{a^2}{4} - b} = M$, the solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{M-1} \mathrm{e}^{-at/2} \left[C_n \sinh(z_n t) + D_n \cosh(z_n t)\right] \sin(n\pi x/L) \\ &+ \mathrm{e}^{-at/2}(E_M t + F_M)\sin(M\pi x/L) \\ &+ \sum_{n=M+1}^{\infty} \mathrm{e}^{-at/2} \left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L), \end{aligned}$$

with the coefficients:

$$E_M = \frac{aF_M}{2}, \quad F_M = \frac{2}{L}\int_0^L f(s)\sin(M\pi s/L)\,ds.$$

Problem 5.2.7

a) $\lambda_{n,m} = \dfrac{n^2\pi^2}{L^2} + \dfrac{m^2\pi^2}{H^2}$, $n \geq 0$, $m \geq 1$, $\varphi_{n,m}(x,y) = \cos(n\pi x/L)\sin(m\pi y/H)$.

b) If $L = H$, we have $\lambda_{n,m} = \lambda_{m,n}$, while $\varphi_{n,m} \neq \varphi_{m,n}$ for every $n \neq m$. If $L = 2H$ the same happens with the pairs (n, m) and $(2m, n/2)$ for every $n \neq m$.

c)

$$\int_0^L \int_0^H \varphi_{n,m}\, \varphi_{n',m'}\, dydx = \int_0^L \cos(\frac{n\pi x}{L}) \cos(\frac{n'\pi x}{L})\, dx \cdot \int_0^H \sin(\frac{m\pi y}{H}) \sin(\frac{m'\pi y}{H})\, dy,$$

that is zero always that $n \neq n'$ or $m \neq m'$.

Problem 5.2.8

a) We separate variables $u(\vec{x},t) = X(\vec{x})T(t)$ and obtain the problems

$$\begin{cases} \Delta X + \dfrac{\lambda}{c^2} X = 0, & \Omega, \\ X = 0, & \partial\Omega; \end{cases} \qquad T'' + \lambda T = 0.$$

Let (ϕ, λ) and (ψ, μ) be two pairs of eigenfunctions and eigenvalues. The corresponding equations are $\Delta\phi + \dfrac{\lambda}{c^2}\phi = 0$, $\Delta\psi + \dfrac{\mu}{c^2}\psi = 0$, Multiplying the first one by ψ and the second by ϕ, subtracting and integrating on Ω we obtain, using the boundary conditions:

$$0 = \int_\Omega (\psi\Delta\phi - \phi\Delta\psi) = (\lambda - \mu) \int_\Omega \phi\psi \frac{1}{c^2}.$$

Then, if $\lambda \neq \mu$ we have that ϕ and ψ are orthogonal with respect to the weight $\sigma(\vec{x}) = \dfrac{1}{c^2(\vec{x})}$.

b) Using the Rayleigh quotient, we have for the pair (λ, φ) that $\lambda = \dfrac{c^2 \displaystyle\int |\nabla\varphi|^2}{\displaystyle\int |\varphi|^2} \geq 0$. If $\lambda = 0$ the corresponding eigenfunction verifies $\displaystyle\int |\nabla\varphi|^2 = 0$, that is, it is constant; but the boundary condition implies $\varphi = 0$.

c) Let (ϕ_n, λ_n), $n \geq 1$, the pairs of eigenfunctions and eigenvalues. Solving the time equation implies that the solution has the form

$$u(\vec{x},t) = \sum_{n=1}^{\infty} [A_n \sin(\omega_n t) + B_n \cos(\omega_n t)]\phi_n(\vec{x}),$$

where the frequencies of vibration are $\omega_n = \sqrt{\lambda_n}$.

Problem 5.2.9

a) The eigenvalues obtained when we separate variables are $\lambda_{n,m} = \dfrac{\eta_{2n,m}^2}{a^2}$, $n,\ m \geq 1$, where $\eta_{2n,m} = m$–th zero of the Bessel function J_{2n}. Then, the vibrating frequencies are $\omega_{n,m} = \dfrac{c\eta_{2n,m}}{a}$.

b)

$$u(r,\theta,t) = \sum_{n=1} \sum_{m=1} [A_{n,m} \sin(\omega_{n,m} t) + B_{n,m} \cos(\omega_{n,m} t)] \sin(2n\theta) J_{2n}(\eta_{2n,m} r/a),$$

where the coefficients are $A_{n,m} = 0$ for every m, $m \geq 1$,

$$B_{n,m} = \frac{4\displaystyle\int_0^{\pi/2}\int_0^a g(r,\theta)\sin(2n\theta)J_{2n}(\eta_{2n,m}r/a)r\,drd\theta}{\pi\displaystyle\int_0^a J_{2n}^2(\eta_{2n,m}r/a)r\,dr}.$$

Problem 5.2.10 Since the equation and the data are independent of θ and the domain is invariant under rotations around the vertical axis, z, the solution will not depend on θ. Then, we separate variables in the form (r, z, t) and the solution is

$$u(r,z,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m}\sin(n\pi z/H)J_0(\eta_{0,m}r/a)\mathrm{e}^{-\lambda_{n,m}kt},$$

where $\lambda_{n,m} = \dfrac{n^2\pi^2}{H^2} + \dfrac{\eta_{0,m}^2}{a^2}$, $\eta_{0,m}$ is the m–th zero of the Bessel function J_0, and the coefficients are

$$a_{n,m} = \frac{\dfrac{2}{H}\displaystyle\int_0^H\int_0^a f(r,z)\sin(n\pi z/H)J_0(\eta_{0,m}r/a)\,rdrdz}{\displaystyle\int_0^a J_0^2(\eta_{0,m}r/a)\,rdr}.$$

Problem 5.2.11 We separate variables in (r, θ, z, t) and we only have to be careful when constant solutions appear in some cases by the Newmann condition at the boundary. The solution is:

$$u(r,\theta,z,t) = \sum_{n=0}^{\infty}\cos(n\pi z/H)\Big[A_{n,0,0}\mathrm{e}^{-\lambda_{n,0,0}kt} + \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} A_{n,m,p}\cos(2m\theta)J_{2m}(\nu_{2m,p}r/a)\mathrm{e}^{-\lambda_{n,m,p}kt}\Big],$$

where

$$\lambda_{n,m,p} = \begin{cases} \dfrac{n^2\pi^2}{H^2} + 4m^2 + \dfrac{\nu_{2m,p}^2}{a^2}, & n \geq 0,\ m \geq 0,\ p \geq 1, \\ \dfrac{n^2\pi^2}{H^2}, & n \geq 0,\ m = p = 0, \end{cases}$$

and $\nu_{2m,p}$ is the p–th zero of the Bessel function J'_{2m}. Clearly

$$\lim_{t\to\infty} u(r,\theta,z,t) = A_{0,0,0} = \frac{1}{|\Omega|}\int_\Omega f = \frac{4}{\pi a^2 H}\int_0^H\int_0^{\pi/2}\int_0^a f(r,\theta,z)r\,drd\theta dz.$$

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