uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS. Solutions

Degree in Biomedical Engineering

Chapter 6

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6 Fourier transform

6.1 Basic properties

Problem 6.1.1 *a)* $\int_{-\infty}^{\infty} g(x)\,dx = \alpha \int_{-\infty}^{\infty} f(\alpha x)\,dx = \int_{-\infty}^{\infty} f(y)\,dy = 1;$

b) $\widehat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha f(\alpha x) \mathrm{e}^{i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\mathrm{e}^{i\omega y/\alpha}\,dy = \widehat{f}(\omega/\alpha);$

c) $\int_{\mathbb{R}^n} \beta f(\alpha\vec{x})\,d\vec{x} = \beta\alpha^{-n}\int_{\mathbb{R}^n} f(\vec{y})\,d\vec{y} = 1$ si $\beta = \alpha^n;$

b) $\widehat{g}(\omega) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \alpha^n f(\alpha\vec{x})\mathrm{e}^{i\vec{\omega}\cdot\vec{x}}\,d\vec{x} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\vec{y})\mathrm{e}^{i\vec{\omega}\cdot\vec{y}/\alpha}\,d\vec{y} = \widehat{f}(\vec{\omega}/\alpha).$

Problem 6.1.2 *a)* $\int_{-\infty}^{\infty} \alpha\, e^{-\beta(x-x_0)^2}\,dx = \frac{\alpha}{\sqrt{\beta}}\int_{-\infty}^{\infty} e^{-z^2}\,dz = \frac{\alpha}{\sqrt{\beta}}\sqrt{\pi} = 1$ si $\alpha = \sqrt{\frac{\beta}{\pi}};$

b) The limit $\lim_{\beta\to\infty}\sqrt{\frac{\beta}{\pi}}\,e^{-\beta(x-x_0)^2} = 0$ is obvious since the exponential dominates the power $\sqrt{\beta}$, always that $x \neq x_0$;

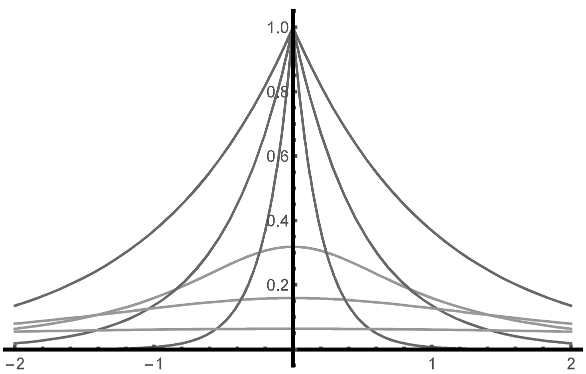
c) $\widehat{g}(\omega) = \sqrt{\frac{\beta}{\pi}}\mathrm{e}^{i\omega x_0}\mathcal{F}(e^{-\beta x^2})(\omega) = \frac{1}{2\pi}e^{-\frac{\omega^2}{4\beta}+i\omega x_0} \longrightarrow \frac{1}{2\pi}e^{i\omega x_0};$

d) $\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(x-x_0)\mathrm{e}^{i\omega x}\,dx = \frac{1}{2\pi}e^{i\omega x_0};$

e) $\mathcal{K}(x,t) = \frac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-\frac{x^2}{4kt}}$, verifies parts *a)* and *b)* with $\beta = \frac{1}{4kt} \to 0.$

Problem 6.1.3 $\widehat{f}(\omega) = \frac{1}{2\pi}\int_{-1}^{1}\mathrm{e}^{i\omega x}\,dx = \frac{\mathrm{e}^{i\omega} - \mathrm{e}^{-i\omega}}{2\pi i\omega} = \frac{\sin\omega}{\pi\omega}.$

Problem 6.1.4 $\widehat{f}(\omega) = \frac{1}{2\pi}\left(\int_{-\infty}^{0}\mathrm{e}^{\alpha x}\mathrm{e}^{i\omega x}\,dx + \int_{0}^{\infty}\mathrm{e}^{-\alpha x}\mathrm{e}^{i\omega x}\,dx\right) = \frac{1}{2\pi}\left(\frac{1}{\alpha+i\omega} - \frac{1}{-\alpha+i\omega}\right) = \frac{\alpha}{\pi(\omega^2+\alpha^2)}.$



Problem 6.1.5 $\mathcal{F}(\Delta f)(\vec{\omega}) = \mathcal{F}\left(\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_j^2}\right)(\vec{\omega}) = \sum_{j=1}^{n}(-i\omega_j)^2\widehat{f}(\vec{\omega}) = -|\vec{\omega}|^2\widehat{f}(\vec{\omega}).$

Problem 6.1.6 By the definition of convolution, $g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)h(-y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(y)|^2\,dy;$

by the definition of the inverse Fourier transform and the Fourier transform of a convolution, $g(0) = \int_{-\infty}^{\infty} \widehat{g}(\omega)\, d\omega = \int_{-\infty}^{\infty} \widehat{f}(\omega)\widehat{h}(\omega)\, d\omega = \int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{f}(\omega)}\, d\omega = \int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\, d\omega.$

Problem 6.1.7 We compute the Fourier transform, $\mathcal{F}(-u''+u)(\omega) = (|\omega|^2+1)\widehat{u}(\omega) = (|\omega|^2+1)\widehat{g}(\omega)\widehat{f}(\omega) = \widehat{f}(\omega)$; so $-u''+u=f$.

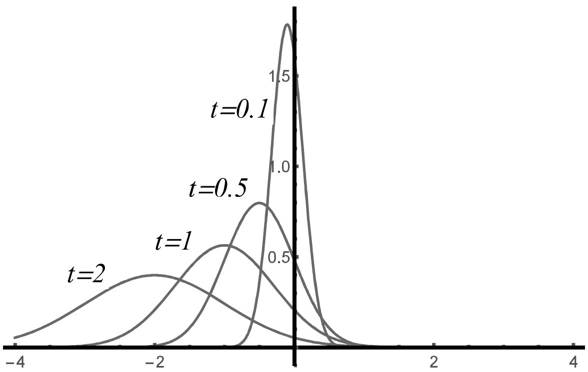
6.2 Resolution of equations

Problem 6.2.1 $\mathcal{F}(-u''+\lambda^2 u)(\omega) = \widehat{f}(\omega) \rightsquigarrow (|\omega|^2+\lambda^2)\widehat{u}(\omega) = \widehat{f}(\omega) \rightsquigarrow \widehat{u}(\omega) = \dfrac{\widehat{f}(\omega)}{|\omega|^2+\lambda^2}$; so $u = f * g$ where $\widehat{g}(\omega) = \dfrac{1}{|\omega|^2+\lambda^2}$; this implies $g(x) = \dfrac{\pi}{\lambda}\mathrm{e}^{-\lambda|x|}$. The solution is finally $u(x) = \dfrac{1}{2\lambda}\displaystyle\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-\lambda|x-y|}\, dy.$

Problem 6.2.2 *a*) We transform by Fourier and obtain $\begin{cases} \dfrac{\partial \widehat{u}}{\partial t} = -k\omega^2\widehat{u} - ci\omega\widehat{u}, & t>0, \\ \widehat{u} = \widehat{f}, & t=0. \end{cases}$,

which gives $\widehat{u}(\omega,t) = \widehat{f}(\omega)\,\mathrm{e}^{-(k\omega^2+ci\omega)t} \rightsquigarrow u(x,t) = f*K(x+ct,t) = \dfrac{1}{\sqrt{4\pi kt}}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x+ct-y)^2}{4kt}}\, f(y)\, dy.$

b) $u(x,t) = \dfrac{1}{\sqrt{4\pi kt}}\mathrm{e}^{-\frac{(x+ct)^2}{4kt}}$, which is a gaussian that as it is re-scaled with the time, lowering and extending itself, it is moving to the left, if $c>0$, with a maximum at $x=-ct$.



c) If we write $u(x,t) = v(x+ct,t)$, then v satisfies the heat equation.

Problem 6.2.3 *a*) We transform the problem and obtain $\begin{cases} \dfrac{\partial \widehat{u}}{\partial t} = -k\omega^2\widehat{u} - \gamma\widehat{u}, & t>0, \\ \widehat{u} = \widehat{f}, & t=0, \end{cases}$

that gives $\widehat{u}(\omega,t) = \widehat{f}(\omega)\,\mathrm{e}^{-(k\omega^2+\gamma)t} \rightsquigarrow u(x,t) = \mathrm{e}^{-\gamma t} f*K(x,t) = \dfrac{\mathrm{e}^{-\gamma t}}{\sqrt{4\pi kt}}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4kt}}\, f(y)\, dy.$

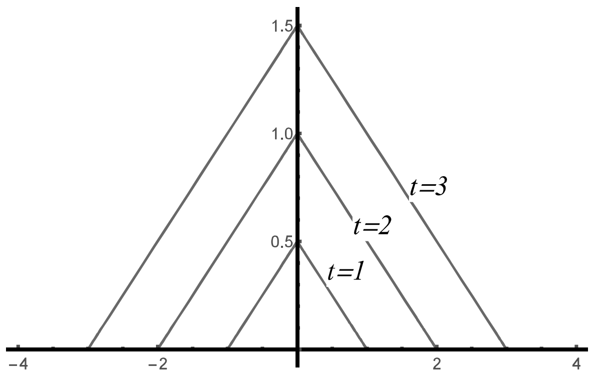
b) If we write $u(x,t) = \mathrm{e}^{-\gamma t}v(x,t)$, then v satisfies the heat equation.

Problem 6.2.4 *a*)

$$\begin{aligned} \widehat{f}(\omega) &= \frac{1}{2\pi}\left(\int_{-a}^{0}(a+x)\mathrm{e}^{i\omega x}\, dx + \int_{0}^{a}(a-x)\mathrm{e}^{i\omega x}\, dx\right) \\ &= \frac{1}{2\pi}\int_0^a (a-x)\left(\mathrm{e}^{i\omega x}+\mathrm{e}^{-i\omega x}\right)\, dx = \frac{1}{\pi}\int_0^a (a-x)\cos\omega x\, dx = \frac{1-\cos a\omega}{\pi\omega^2}. \end{aligned}$$

b) We transform and arrive to

$$\begin{cases} \dfrac{\partial^2 \widehat{u}}{\partial t^2} = -\omega^2 \widehat{u} + \dfrac{1}{2\pi}, & t > 0, \\ \widehat{u} = \dfrac{\partial \widehat{u}}{\partial t} = 0, & t = 0, \end{cases} \quad \text{that implies } \widehat{u}(\omega, t) = \frac{1 - \cos \omega t}{2\pi\omega^2} \rightsquigarrow u(x,t) = \frac{1}{2}(t - |x|)_+.$$



Problem 6.2.5 We transform and obtain

$$\begin{cases} \dfrac{\partial^2 \widehat{u}}{\partial x^2} - \omega^2 \widehat{u} = 0, & 0 < x < L, \\ \widehat{u} = \widehat{g}_1, & x = 0, \\ \widehat{u} = \widehat{g}_2, & x = L, \end{cases} \quad \text{which implies } \widehat{u}(x, \omega) = \widehat{g}_1(\omega) \frac{\sinh((L-x)\omega)}{\sinh(L\omega)} + \widehat{g}_2(\omega) \frac{\sinh(x\omega)}{\sinh(L\omega)}.$$

The solution is then $u = g_1 * F_1 + g_2 * F_2$, where F_1 and F_2 are the inverse Fourier transforms of $\dfrac{\sinh((L-x)\omega)}{\sinh(L\omega)}$ and $\dfrac{\sinh(x\omega)}{\sinh(L\omega)}$ respectively.

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