Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
SELFEVALUATION I - SOLUTIONS
24th to 31st of October, 2017
Degree in Biomedical Engineering

Time: 1 hour
The marking is only valid to check the learning pace. It does not compute for the final note.

Problem 1 (**2 points**)

Solve the equation: $xy' \sin \frac{y}{x} = x + y \sin \frac{y}{x}$.

Solution:
It is homogeneous, because y' is a homogeneous function of degree zero:

$$y' = \frac{1}{\sin \frac{y}{x}} + \frac{y}{x}.$$

We make the change: $z = \frac{y}{x}$, $y' = z + x\frac{dz}{dx}$:

$$z + x\frac{dz}{dx} = \frac{1}{\sin z} + z \quad \Longrightarrow \quad \sin z dz = \frac{dx}{x} \quad \Longrightarrow \quad -\cos z = \log Cx\,.$$

The constant absorbs the sign. Now we undo the change:

$$\cos \frac{y}{x} = \log \frac{K}{x} \quad \Longrightarrow \quad y = x \arccos\Big(\log \frac{K}{x}\Big).$$

Problem 2 (**2 points**)

Solve, using an integrating factor that depends only on one variable:

$$y^2 \cos(x)dx + (4 + 5y\sin(x))dy = 0.$$

Solution:
There is not any integrating factor depending on the variable x, so we look for $\mu = \mu(y)$:

$$\mu(y)y^2 \cos(x)dx + \mu(y)(4 + 5y\sin(x))dy = 0.$$

$$\frac{\partial}{\partial y}(\mu(y)y^2\cos(x)) = \frac{\partial}{\partial x}(\mu(y)(4+5y\sin(x))) \quad \Longrightarrow \quad \cos(x)\Big(\mu'(y)y^2 + 2y\mu(y)\Big) = 5y\cos(x)\mu(y)$$

$$\Longrightarrow \quad \frac{3}{y} = \frac{\mu'(y)}{\mu(y)} \quad \Longrightarrow \quad \log y^3 = \log \mu(y) \quad \Longrightarrow \quad \mu(y) = y^3.$$

We have now an exact equation:

$$y^5 \cos(x)dx + (4y^3 + 5y^4 \sin(x))dy = 0.$$

the solution is $f(x,y)=K$, where:

$$\left\{\begin{array}{l} \dfrac{\partial f}{\partial x}=y^5\cos(x) \quad\Longrightarrow\quad f(x,y)=y^5\sin(x)+C(y) \\ \\ \dfrac{\partial f}{\partial y}=4y^3+5y^4\sin(x) \quad\Longrightarrow\quad C'(y)=4y^3 \quad\Longrightarrow\quad C(y)=y^4 \end{array}\right. \quad\Longrightarrow\quad y^5\sin(x)+y^4=K.$$

Problem 3 (**2 points**)

Solve by reduction of order: $yy''+(y')^2=0$.

SOLUTION:
It is not linear. There is no x, so we reduce the order with the change:

$$y'=p(y) \quad\Longrightarrow\quad y''=p\frac{dp}{dy}=pp'$$

We obtain the equation:

$$ypp'+p^2=0.$$

When $p=0$ then $y'=0 \quad\Longrightarrow\quad y=C$. When $p\neq 0$:

$$yp'+p=0 \quad\Longrightarrow\quad \frac{dp}{p}=-\frac{dy}{y} \quad\Longrightarrow\quad \log|p|=\log\frac{C_1}{|y|} \quad\Longrightarrow\quad p=\frac{C_1}{y}$$

The constant absorbs the two signs. Finally:

$$p=\frac{dy}{dx}=\frac{C_1}{y} \quad\Longrightarrow\quad ydy=C_1dx \quad\Longrightarrow\quad \frac{y^2}{2}=C_1x+C_2 \quad\Longrightarrow\quad y=\pm\sqrt{K_1x+K_2}.$$

Observe that this solution includes $y=C$.

Problem 4 (**2 points**)

Solve the equation: $x^3y'''+xy'-y=3x^4$.

SOLUTION:
This is a nonhomogeneous Euler equation. First we solve the homogeneous part looking for solutions $y=x^r$, and obtain:

$$r(r-1)(r-2)+r-1=0 \quad\Longrightarrow\quad r^3-3r^2+3r-1=0 \quad\Longrightarrow\quad r=1 \text{ (triple)}.$$

We only find one solution, so we use the change of variables $x=e^t, \quad t=\log x$:

$$y_x=y_t\frac{1}{x}, \quad\Longrightarrow\quad y_{xx}=\frac{1}{x^2}(y_{tt}-y_t) \quad\Longrightarrow\quad y_{xxx}=\frac{1}{x^3}(y_{ttt}-3y_{tt}+2y_t).$$

The equation is transformed into another with constant coefficients:

$$x^3\frac{1}{x^3}(y_{ttt}-3y_{tt}+2y_t)+x\frac{1}{x}y_t-y=0 \quad\Longrightarrow\quad y_{ttt}-3y_{tt}+3y_t-y=0.$$

Now we look for solutions $y=e^{Kt} \quad\Longrightarrow\quad K=1$ is a root of order three so:

$$y=e^t\left(C_1+C_2t+C_2t^2\right) \quad\Longrightarrow\quad y_h(x)=x\left(C_1+C_2\log x+C_3(\log x)^2\right).$$

We try now a particular solution $y_p=Ax^4$ and obtain $y_p=\dfrac{x^4}{9}$. Finally:

$$y(x)=y_p+y_h=\frac{x^4}{9}+x\left(C_1+C_2\log x+C_3(\log x)^2\right).$$

Problem 5 (**2 points**)
Use Laplace transform to solve for $\omega \neq \omega_0$ the initial value problem

$$\begin{cases} x'' + \omega_0^2\, x = k \sin \omega t, & t > 0 \\ \\ x(0) = x'(0) = 0 \end{cases}$$

which describes the forced oscillations of a mass on an undamped spring. What happens if $\omega = \omega_0$?
Hint: Verify that $L^{-1}\{\frac{1}{(s^2+a^2)^2}\} = \frac{1}{2a^3}(\sin at - at\cos at)$.

SOLUTION:
We transform by Laplace the equation and obtain:

$$(s^2 + \omega_0^2)L[x](s) = \frac{k\omega}{s^2 + \omega^2} \qquad \Longrightarrow \qquad L[x](s) = \frac{k\omega}{(s^2 + \omega^2)(s^2 + \omega_0^2)}$$

Now, we decompose:

$$L[x](s) = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + \omega_0^2} = \frac{kw}{\omega_0^2 - \omega^2}\left\{\frac{1}{s^2 + \omega^2} + \frac{1}{s^2 + \omega_0^2}\right\}.$$

We antitransform and obtain:

$$x(t) = \frac{k}{\omega_0} \cdot \frac{\omega_0 \sin \omega t - \omega \sin \omega_0 t}{\omega_0^2 - \omega^2}.$$

If now $\omega = \omega_0$, then:

$$L[x](s) = \frac{k\omega_0}{(s^2 + \omega_0^2)^2},$$

and using the hint we have that:

$$x(t) = \frac{k}{2\omega_0^2}(\sin \omega_0 t - \omega_0 t \cos \omega_0 t).$$

To prove the hint we can go backwards, using the properties of Laplace transform:

$$L\left[\frac{1}{2a^3}(\sin at - at\cos at)\right](s) = \frac{1}{2a^3}\left(\frac{a}{s^2 + a^2} + \frac{\partial}{\partial s}\frac{as}{s^2 + a^2}\right) = \frac{1}{(s^2 + a^2)^2}.$$