Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
SELF-EVALUATION II - SOLUTIONS
5th to 12th of December, 2017
Degree in Biomedical Engineering.

Time: 90 minutes
The marking is only valid to check the learning pace. It does not compute for the final note.

Problem 1 (**4 points**)

Solve the Laplace equation over a semicircle:

$$\Delta u = 0, \quad \{0 < r < a,\, 0 < \theta < \pi\}$$

with the boundary conditions:
The diameter is isolated and: $u(a,\theta) = g(\theta)$.

Solution:
The Laplace operator in polar coordinates is:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

and the conditions are:

$$u_\theta(r,0) = 0, \quad u_\theta(r,\pi) = 0, \quad u(a,\theta) = g(\theta).$$

Note that we must include the boundedness condition at the origin: $|u(0,\theta)| < \infty$.
Next, we separate variables: $u(r,\theta) = \varphi(\theta)G(r)$. Thus, we have to solve on one side the angular problem:

$$\begin{cases} \varphi''(\theta) + \lambda\varphi(\theta) = 0 \\ \varphi'(0) = 0, \quad \varphi'(\pi) = 0 \end{cases} \implies \begin{cases} \text{eigenvalues} \quad \lambda_n = n^2, \quad n = 0,1,2,\ldots \\ \text{eigenfunctions} \quad \varphi_n(\theta) = \cos n\theta. \end{cases}$$

The other one is the radial problem:

$$\begin{cases} r^2G''(r) + rG(r) - n^2G(r) = 0 \\ |G(0)| < \infty \end{cases} \implies \begin{cases} G_n(r) = Cr^n, \quad n \neq 0 \\ G_0(r) = C, \qquad n = 0. \end{cases}$$

Therefore, the solution to our problem will be:

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n\,(r)^n\,\cos n\theta.$$

Moreover, thanks to the initial condition and the orthogonality of the eigenfunctions, we arrive at:

$$u(r,\theta) = \frac{1}{\pi}\int_0^\pi g(\phi)d\phi + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\,\frac{2}{\pi}\left(\int_0^\pi g(\phi)\cos n\phi\, d\phi\right)\,\cos n\theta\,.$$

Problem 2 (**2 points**)

Use the Rayleigh quotient to obtain a reasonably precise upper bound for the first eigenvalue of the problem:

$$\varphi'' + (\lambda - x)\varphi = 0\,, \qquad \varphi'(0) = 2\varphi(1) + \varphi'(1) = 0.$$

SOLUTION:
Observe we are dealing with a Sturm-Liouville problem with $p = 1$, $q = -x$, $\sigma = 1$. Due to the minimization principle. we look for functions which satisfy the boundary conditions, for example $\phi(x) = x^2 - 2$ and substitute it into the Rayleigh quotient:

$$\lambda_1 \leq \frac{[-\phi\phi']_0^1 + \displaystyle\int_0^1 (\phi')^2 + \int_0^1 x\phi^2}{\displaystyle\int_0^1 \phi^2} = \frac{135}{106}.$$

Problem 3 (**4 points**)
Solve the telegraph problem:

$$\begin{cases} u_{tt} - u_{xx} + au_t + bu = 0 & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0 & t > 0, \\ u(x,0) = f(x) & 0 < x < L, \\ u_t(x,0) = 0 & 0 < x < L. \end{cases}$$

SOLUTION:
Solving the spatial equation (obtained after applying the separate variables method) we have that

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin(\frac{n\pi x}{L}), \quad n \geq 1.$$

On the other hand the temporal equation is as follows

$$T'' + aT' + (b + \lambda)T = 0.$$

The solution is different depending on the values of λ in relation with a and b. Thus, if

$$\frac{\pi^2}{L^2} > \frac{a^2}{4} - b, \quad \left(\text{or} \quad \frac{n\pi^2}{L^2} > \frac{a^2}{4} - b, \quad \text{for any } n \geq 1\right),$$

the solution to our problem will be

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-at/2}\left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L),$$

where $w_n = \sqrt{\dfrac{n^2\pi^2}{L^2} - \dfrac{a^2}{4} + b}$, and with coefficients

$$A_n = \frac{aB_n}{2w_n}, \quad B_n = \frac{2}{L}\int_0^L f(s)\sin(n\pi s/L)\,ds.$$

Moreover, if there exists $M \in \mathbb{N}$ such that $M - 1 < \frac{L}{\pi}\sqrt{\frac{a^2}{4} - b} < M$, the solution will be

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{M-1} \mathrm{e}^{-at/2} \left[C_n \sinh(z_n t) + D_n \cosh(z_n t)\right] \sin(n\pi x/L) \\ &+ \sum_{n=M}^{\infty} \mathrm{e}^{-at/2} \left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L), \end{aligned}$$

where $z_n = \sqrt{\frac{a^2}{4} - b - \frac{n^2\pi^2}{L^2}}$—, and the coefficients for n between 1 and $M-1$ are the following

$$C_n = \frac{aD_n}{2z_n}, \quad D_n = \frac{2}{L}\int_0^L f(s)\sin(n\pi s/L)\,ds.$$

Finally, if there exists $M \in \mathbb{N}$ such that $\frac{L}{\pi}\sqrt{\frac{a^2}{4} - b} = M$, the solution will be

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{M-1} \mathrm{e}^{-at/2} \left[C_n \sinh(z_n t) + D_n \cosh(z_n t)\right] \sin(n\pi x/L) \\ &+ \mathrm{e}^{-at/2}(E_M t + F_M)\sin(M\pi x/L) \\ &+ \sum_{n=M+1}^{\infty} \mathrm{e}^{-at/2} \left[A_n \sin(w_n t) + B_n \cos(w_n t)\right] \sin(n\pi x/L) \end{aligned}$$

with the coefficients as follows

$$E_M = \frac{aF_M}{2}, \quad F_M = \frac{2}{L}\int_0^L f(s)\sin(M\pi s/L)\,ds.$$
