Universidad Carlos III de Madrid

Escuela Politécnica Superior

Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
CONTROL I - SOLUTIONS
7th of November, 2017
Degree in Biomedical Engineering.

Time: 90 minutes

Problem 1 (**2.5 points**)

Solve the equation:

$$(3x^2y^2 - y\cos x)dx = (\sin x - 2x^3y)dy.$$

Solution:
We rearrange and prove that it is an exact differential equation:

$$(3x^2y^2 - y\cos x)dx + (-\sin x + 2x^3y)dy = 0,$$

$$\frac{\partial}{\partial y}(3x^2y^2 - y\cos x) = 6x^2y - \cos x = \frac{\partial}{\partial x}(-\sin x + 2x^3y),$$

whose solution is $f(x,y) = K$ where

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3x^2y^2 - y\cos x \quad \Longrightarrow \quad f(x,y) = x^3y^2 - y\sin x + C(y) \\ \dfrac{\partial f}{\partial y} = -\sin x + 2x^3y = 2x^3y - \sin x + C'(y) \quad \Longrightarrow \quad C(y) = c \end{cases}$$

Solution: $x^3y^2 - y\sin x = K$.

Problem 2 (**2.5 points**)

Find the solution of:

$$x^4yy' + \left(\frac{3}{2}y^2 + 1\right)x^3 = 1\ .$$

Solution:
This is a Bernoulli equation:

$$y' + \frac{3}{2x}y = \left(\frac{1}{x^4} - \frac{1}{x}\right)\frac{1}{y},$$

with $n = -1$, so we change:

$$z = y^{1-n} = y^2 \implies z' = 2yy' \implies y' = \frac{z'}{2y},$$

and find the new equation, that is linear:

$$\frac{z'}{2y} + \frac{3}{2x}y = \left(\frac{1}{x^4} - \frac{1}{x}\right)\frac{1}{y},$$

$$z' + \frac{3}{x} z = \frac{2}{x^4} - \frac{2}{x}$$

We solve z and undo the change:

$$z = \mathrm{e}^{-\int \frac{3}{x} dx} \left[\int \left(\frac{2}{x^4} - \frac{2}{x} \right) \mathrm{e}^{\int \frac{3}{x} dx} dx + C \right] = \frac{1}{x^3} \left[\int \left(\frac{2}{x} - 2x^2 \right) dx + C \right] = \frac{\log(x^2) + C}{x^3} - \frac{2}{3},$$

$$y = \pm \sqrt{\frac{\log(x^2) + C}{x^3} - \frac{2}{3}}.$$

Also, the equation can be solved with the integrating factor $\mu(x) = \dfrac{1}{x}$.

Problem 3 (**2.5 points**)

Solve the following equation:

$$xy'' + 3y' + \frac{1}{x} y = 4x^2.$$

SOLUTION:
Multiplying the equation by x we obtain

$$x^2 y'' + 3xy' + y = 4x^3,$$

which is actually a non-homogenous Euler-type equation. First we solve the homogenous part performing the change of variable:

$$x = e^t, \quad t = \log x \quad \Longrightarrow \quad y_x = y_t \frac{1}{x}, \quad y_{xx} = \frac{1}{x^2}(y_{tt} - y_t).$$

We find the new equation:

$$x^2 \frac{1}{x^2}(y_{tt} - y_t) + 3xy_t \frac{1}{x} + y = 0 \quad \Longrightarrow \quad y_{tt} + 2y_t + y = 0.$$

The new equation has constant coefficients, so we try $y = \mathrm{e}^{rt}$ as a solution:

$$y = \mathrm{e}^{rt} \quad \Longrightarrow \quad r^2 + 2r + 1 = 0 \quad \Longrightarrow \quad r = -1 \quad \text{(double root)}$$

and hence,

$$y = \mathrm{e}^{-t}(c_1 + c_2 t) \quad \Longrightarrow \quad y_h(x) = \frac{1}{x}(c_1 + c_2 \log x).$$

Now, we look for a particular solution of the form $y_p = Ax^3$, we calculate

$$y_p' = 3Ax^2, \quad y_p'' = 6Ax,$$

and we find

$$Ax^3 \Big[6 + 9 + 1 \Big] = 4x^3 \quad \Longrightarrow \quad A = \frac{4}{16} \quad \Longrightarrow \quad y_p = \frac{1}{4} x^3.$$

Consequently, the general solution of the original equation will be the following:

$$y = y_h + y_p = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{4} x^3.$$

Problem 4 (**2.5 points**)

Solve the problem:

$$\begin{cases} x'' - 2x' + x = f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ 0, & 2 \le t, \end{cases} \\ x(0) = x'(0) = 0\,; \end{cases}$$

Solution:

We use the Laplace transform and first we rewrite the second term using the Heavyside function:

$$f(t) = e^t - H(t-2)e^t = e^t - H(t-2)e^{(t-2)}e^2$$

Applying the Laplace transformation to the differential equation it follows that

$$s^2 L[x](s) - 2sL[x](s) + L[x](s) = (s-1)^2 L[x](s) = \frac{1}{s-1} - \frac{e^{-2s}e^2}{s-1}$$

Now we antitransform:

$$L[x](s) = \frac{1}{(s-1)^3} - \frac{e^{-2s}e^2}{(s-1)^3}$$

$$\Longrightarrow x(t) = \frac{t^2 e^t}{2} - H(t-2)\frac{e^2(t-2)^2 e^{t-2}}{2} = \begin{cases} \dfrac{t^2 e^t}{2}, & 0 < t < 2, \\ e^t(2t-2), & 2 \le t. \end{cases}$$