Universidad Carlos III de Madrid
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Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
CONTROL 2 - SOLUTIONS

18th of December 2017
Degrees in Biomedical Engineering.

Time: 3 hours

Problem 1 (**4 points**)
Solve the following problem on a rectangle:

$$\begin{cases} u_{xx}+4u_{yy}=0, & 0<x<\pi/2, \quad 0<y<\pi, \\ u(0,y)=u(\pi/2,y)=0, & 0<y<\pi, \\ u(x,\pi)=0, & 0<x<\pi/2, \\ u(x,0)=2\sin 6x, & 0<x<\pi/2\,. \end{cases}$$

Solution:
We look for solutions of the form: $u(x,y)=\Phi(x)G(y)$:

$$\Phi''(x)G(y)+4\Phi(x)G''(y)=0 \quad\Longrightarrow\quad \frac{\Phi''(x)}{\Phi(x)}=-\frac{4G''(y)}{G(y)}=-\lambda\,.$$

The eigenvalue problem is:

$$\begin{cases} \Phi''(x)+\lambda\Phi(x)=0 \quad\Longrightarrow\quad G(x)=e^{rx}, \quad r^2+\lambda=0 \quad\Longrightarrow\quad r=\pm\sqrt{-\lambda}\,, \\ \Phi(0)=\Phi(\frac{\pi}{2})=0\,, \end{cases}$$

Case $\lambda>0: r=\pm i\sqrt{\lambda} \quad\Longrightarrow\quad \Phi(x)=c_1\cos(\sqrt{\lambda}\,x)+c_2\sin(\sqrt{\lambda}\,x)$.

$$\Phi(0)=c_1=0, \quad \Phi(\frac{\pi}{2})=c_2\sin(\sqrt{\lambda}\,\frac{\pi}{2})=0 \quad\Longrightarrow\quad \sqrt{\lambda}\,\frac{\pi}{2}=n\pi \quad\Longrightarrow\quad \lambda=4n^2,\ n=1,2,\ldots$$

These eigenvalues correspond to the eigenfunctions: $\Phi_n(x)=\sin(2nx)$, with $n=1,2,\ldots$.
Case $\lambda=0$: $r=0 \quad\Longrightarrow\quad G(x)=c_1+c_2x$.

$$\Phi(0)=c_1=0, \quad \Phi(\frac{\pi}{2})=c_1+c_2\frac{\pi}{2}=c_2\frac{\pi}{2}=0 \quad\Longrightarrow\quad c_2=0.$$

So this is not an eigenvalue.
Case $\lambda<0: r=\pm\sqrt{\lambda} \quad\Longrightarrow\quad \Phi(x)=c_1\cosh(\sqrt{-\lambda}\,x)+c_2\sinh(\sqrt{-\lambda}\,x)$.

$$\Phi(0)=c_1=0, \quad \Phi(\frac{\pi}{2})=c_2\sinh(\sqrt{-\lambda}\,\frac{\pi}{2})=0 \quad\Longrightarrow\quad c_2=0\,.$$

It is not an eigenvalue either.
Problem for y (now $\lambda=4n^2$):

$$\begin{cases} 4G''(y)-4n^2G(x)=0 \quad\Longrightarrow\quad G(x)=e^{ry}, \quad 4r^2-4n^2=0 \quad\Longrightarrow\quad r=\pm n\,, \\ G(\pi)=0\,, \end{cases}$$

We can write $G_n(y)=c_1e^{ny}+c_2e^{-ny}$, but it is simpler to use:

$$G_n(y)=c_1\cosh\Big(n(y-\pi)\Big)+c_2\sinh\Big(n(y-\pi)\Big).$$

Since $G_n(\pi) = c_1 = 0 \quad \Longrightarrow \quad G_n(y) = c_2 \sinh\Big(n(y-\pi)\Big)$.
Now we obtain the product solution and also apply the superposition principle:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(2nx)\, \sinh\Big(n(y-\pi)\Big)\,,$$

With the boundary condition at $y = 0$ we obtain the coefficients:

$$u(x,0) = 2\sin(6x) = \sum_{n=1}^{\infty} B_n \sin(2nx)\, \sinh(-n\pi)$$
$$\Longrightarrow B_n = 0, \quad n \neq 3\,, \quad B_3 = \frac{-2}{\sinh(3\pi)}$$

Solution:

$$u(x,y) = \frac{-2}{\sinh(3\pi)} \sin(6x) \sinh\Big(3(y-\pi)\Big)\,.$$

Problem 2 (**1 + 2 points**)
Consider the Laplace problem in a disc with radius 3:

$$\begin{cases} \Delta u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 3, \\ \\ u(3,\theta) = f(\theta). \end{cases}$$

a) Apply the method of separate variables and find the one variable problems.

b) Solve those problems (ODE's).

SOLUTION:

a) First we must add the conditions of periodicity and boundedness at the origin:

$$u(r,\pi) = u(r,-\pi), \qquad u_\theta(r,\pi) = u_\theta(r,-\pi), \qquad |u(0,\theta)| < \infty.$$

Applying the method of separation of variables

$$u(r,\theta) = \phi(\theta)G(r),$$

we obtain the problems:

$$\begin{cases} \phi''(\theta) + \lambda\varphi(\theta) = 0, \\ \phi(\pi) = \phi(-\pi), \\ \phi'(-\pi) = \phi'(-\pi). \end{cases} \qquad \begin{cases} r^2G''(r) + rG(r) - \lambda G = 0, \\ |G(0)| < \infty. \end{cases}$$

b) The angular problem provides us with a family of eigenvalues and eigenfunctions:

$$\begin{array}{lll} \lambda_0 = 0, & & \phi_0 = 1, \\ \lambda_n = n^2, & n = 1,2,\ldots & \phi_n = C_1\cos(n\theta) + C_2\sin(n\theta)\,. \end{array}$$

Now, we solve the radial problem, which is an Euler equation. Substituting $\lambda = n^2$ into the radial equation and separating the cases $n = 0$ and $n \neq 0$, we find that, after applying the boundedness condition at the origin,

$$\begin{cases} n = 0: & r^2G''(r) + rG'(r) = 0 & \Longrightarrow & G(r) = C_1 + C_2\log r & \Longrightarrow & G_0(r) = C_1. \\ n \neq 0: & r^2G''(r) + rG'(r) - n^2G(r) = 0 & \Longrightarrow & G(r) = C_1r^n + C_2r^{-n} & \Longrightarrow & G_n(r) = C_1r^n. \end{cases}$$

Problem 3 (**1,5 + 1,5 points**)
Consider the problem:

$$\begin{cases} \varphi'' + 2\varphi' + (\lambda - x)\varphi = 0, & 0 < x < 1, \\ \varphi'(0) = \varphi(1) = 0. \end{cases}$$

a) Write it in form of a Sturm-Liuville problem using an integrating factor.

b) Study if all the eigenvalues are positive and if there is a zero eigenvalue.

SOLUTION:
a) We multiply by a factor $H(x)$, we want to obtain the form of a Sturm-Liouville problem

$$H(\varphi'' + 2\varphi' + (\lambda - x)\varphi) = (p\varphi')' + q\varphi + \lambda\sigma\varphi.$$

Then we must have

$$p = H, \quad p' = 2H, \quad q = -xH, \quad \sigma = H.$$

So, we deduce that:

$$p'/p = 2, \Longrightarrow p = H = e^{2x}.$$

The new equation is

$$(e^{2x}\varphi')' - xe^{2x}\varphi + \lambda e^{2x}\varphi = 0.$$

b) One can obtain the Rayleigh quotient with this steps: we multiply the equation by φ, then integrate on $(0, 1)$:

$$\int_0^1 \left(e^{2x}\varphi'(x)\right)'\varphi(x)\,dx - \int_0^1 xe^{2x}\varphi^2(x)\,dx + \lambda \int_0^1 e^{2x}\varphi^2(x)\,dx = 0,$$

and finally we clear λ and integrate by parts the first integral, making the substitution of the boundary conditions,

$$\lambda = \frac{\left[-e^{2x}\varphi(x)\varphi'(x)\right]_0^1 + \displaystyle\int_0^1 e^{2x}\left(\varphi'(x)\right)^2 dx + \int_0^1 xe^{2x}\varphi^2(x)\,dx}{\displaystyle\int_0^1 e^{2x}\varphi^2(x)\,dx}$$

$$= \frac{\displaystyle\int_0^1 e^{2x}\left(\varphi'(x)\right)^2 dx + \int_0^1 xe^{2x}\varphi^2(x)\,dx}{\displaystyle\int_0^1 e^{2x}\varphi^2(x)\,dx}$$

Everything is greater or equal to zero, and the denominator is never zero, so $\lambda \geq 0$. Also, if $\lambda = 0$ then all the terms in the numerator must be zero, in particular

$$\int_0^1 xe^{2x}\varphi^2(x)\,dx \;\Rightarrow\; \varphi \equiv 0,$$

so $\lambda = 0$ is not an eigenvalue and all the eigenvalues are strictly positive: $\lambda > 0$.