uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
EXTRAORDINARY EXAM - SOLUTIONS
18th of June, 2018
Degree in Biomedical Engineering.

Time: 3 hours

Problem 1 (**1.5 points**)

Solve the equation $\quad 3xy^2y' + y^3 = x\sin x.$

SOLUTION:
We divide by $3xy^2$ and obtain a Bernouilli equation with $n = -2$:

$$y' + \frac{y}{3x} = \frac{\sin x}{3} y^{-2},$$

Now we change $z = y^{1-n} = y^3 \implies z' = 3y^2y' \implies y' = \dfrac{z'}{3y^2}$ and obtain a linear equation:

$$\frac{z'}{3y^2} + \frac{y}{3x} = \frac{\sin x}{3} y^{-2} \implies z' + \frac{z}{x} = \sin x.$$

Applying the formula for the solution we arrive to:

$$\begin{aligned} z &= e^{-\int \frac{dx}{x}} \left[\int \sin x\, e^{\int \frac{dx}{x}} dx + C \right] = \frac{1}{x} \left[\int x \sin x\, dx + C \right] \\ &= \frac{1}{x} \left[-x\cos x + \int \cos x\, dx + C \right] = -\cos x + \frac{\sin x + C}{x}, \end{aligned}$$

and finally:

$$y = z^{1/3} = \sqrt[3]{-\cos x + \frac{\sin x + K}{x}}.$$

Problem 2 (**2 points**)

Solve the equation $\quad xy'' + 3y' + \dfrac{1}{x}\, y = x^2.$

SOLUTION:
Multiplying by x we have $\quad x^2y'' + 3xy' + y = x^3$, that is a non-homogeneous Euler type equation. We solve first the homogeneous part, with the change:

$$x = \mathrm{e}^t, \quad t = \log x \quad \implies \quad y_x = y_t \frac{1}{x}, \quad y_{xx} = \frac{1}{x^2}(y_{tt} - y_t).$$

The new equation is:

$$x^2 \frac{1}{x^2}(y_{tt} - y_t) + 3xy_t \frac{1}{x} + y = 0 \quad \implies \quad y_{tt} + 2y_t + y = 0.$$

Making $y = e^{rt} \implies r^2 + 2r + 1 = 0 \implies r = -1$ double root, so:

$$y = e^{-t}(c_1 + c_2 t) \implies y_h(x) = \frac{1}{x}(c_1 + c_2 \log x).$$

Finally we look for $y_p = Ax^3 \implies y_p' = 3Ax^2, \quad y_p'' = 6Ax$:

$$Ax^3\Big[6 + 9 + 1\Big] = x^3 \implies A = \frac{1}{16} \implies y_p = \frac{1}{16}x^3.$$

The whole solution is:

$$y = y_h + y_p = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{16}x^3.$$

Problem 3 (**2 points**)

Solve the initial value problem

$$\begin{cases} y'' + 2y' + 2y = g(t), \\ y(0) = 1, \quad y'(0) = 0, \end{cases} \qquad g(t) = \begin{cases} 3, & 0 < t < 2\pi, \\ 0, & 2\pi \le t < 5\pi, \\ 1, & 5\pi \le t. \end{cases}$$

SOLUTION:
We write $g(t)$ in terms of a Heaviside function and transform by Laplace the equation:

$$\begin{aligned} g(t) &= 3 + H(t-2\pi)(-3) + H(t-5\pi) \\ (s^2+2s+2)L[y](s) - s - 2 &= \frac{3}{s} - \frac{3e^{-2\pi s}}{s} + \frac{e^{-5\pi s}}{s} \\ L[y](s) &= \frac{3+s^2+2s}{s(s^2+2s+2)} + \frac{-3e^{-2\pi s} + e^{-5\pi s}}{s(s^2+2s+2)}. \end{aligned}$$

Now we write this in simple fractions so that we can antitransform:

$$\frac{3+s^2+2s}{s(s^2+2s+2)} = \frac{3/2}{s} - \frac{1}{2}\frac{(s+1)+1}{(s+1)^2+1} = L\left[\frac{3}{2} - \frac{e^{-t}}{2}(\cos t + \sin t)\right](s),$$

$$\frac{1}{s(s^2+2s+2)} = \frac{s}{2} - \frac{\frac{1}{2}(s+1)+\frac{1}{2}}{(s+1)^2+1} = L\left[\frac{1}{2} - \frac{e^{-t}}{2}(\cos t + \sin t)\right](s).$$

The solution is:

$$\begin{aligned} y(t) &= \frac{3}{2} - \frac{e^{-t}}{2}(\cos t + \sin t) - \frac{3}{2}H(t-2\pi)\left(1 - e^{-(t-2\pi)}(\cos(t-2\pi) + \sin(t-2\pi))\right) \\ &\quad + \frac{1}{2}H(t-5\pi)\left(1 - e^{-(t-5\pi)}(\cos(t-5\pi) + \sin(t-5\pi))\right) \\ &= \frac{3}{2} - \frac{e^{-t}}{2}(\cos t + \sin t) - \frac{3}{2}H(t-2\pi)\left(1 - e^{-(t-2\pi)}(\cos t + \sin t)\right) \\ &\quad + \frac{1}{2}H(t-5\pi)\left(1 - e^{-(t-5\pi)}(-\cos t - \sin t)\right). \end{aligned}$$

Problem 4 (**0.5 + 1 + 1 = 2.5 points**)

a) Split into two one-variable problems

$$\begin{cases} u_t - 2u_{xx} = 2u, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad t > 0, \\ u(-\pi/2, t) = u(\pi/2, t), & t > 0, \\ u_x(-\pi/2, t) = u_x(\pi/2, t), & t > 0, \\ u(x, 0) = 8 - 3\sin(2x) - 8\cos^2(2x), & -\frac{\pi}{2} < x < \frac{\pi}{2}. \end{cases}$$

b) Solve both problems.

c) Find the solution of the original problem.

SOLUTION:

a) We separate $u(x,t) = \phi(x)G(t)$ and obtain the problems:

$$\begin{cases} \phi''(x) + \lambda\phi(x) = 0, \\ \phi(-\pi/2) = \phi(\pi/2), \\ \phi'(-\pi/2) = \phi'(\pi/2), \end{cases} \qquad \left\{ G'(t) = 2(1-\lambda)G(t). \right.$$

b) With the boundary conditions we know that (or we calculate them):

$$\lambda_n = \left(\frac{n\pi}{\pi/2}\right)^2 = 4n^2, \quad n = 0, 1, 2, \ldots \qquad \phi_n(x) = c_1 \cos(2nx) + c_2 \sin(2nx)$$

Then

$$G'(t) = 2(1-4n^2)G(t) \quad \Longrightarrow \quad G(t) = Ke^{2(1-4n^2)t}.$$

c) The product solution, using the superposition principle, is:

$$u(x,t) = A_0 \mathrm{e}^{2t} + \sum_{n=1}^{\infty} \mathrm{e}^{(2-8n^2)t} \left(A_n \cos(2nx) + B_n \sin(2nx)\right)$$

The initial condition implies that:

$$\begin{aligned} u(x,0) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(2nx) + B_n \sin(2nx)) &= 8 - 3\sin(2x) - 8\cos^2(2x) \\ &= 8 - 3\sin(2x) - 4(1+\cos(4x)), \end{aligned}$$

and from this we obtain that $B_n = 0, \quad n \neq 1, \quad A_n = 0, \quad n \neq 0, n \neq 2$ y:

$$A_0 = 4, \quad B_1 = -3, \quad A_2 = -4.$$

The solution is thus:

$$u(x,t) = 4e^{2t} - 3e^{-6t}\sin(2x) - 4e^{-30t}\cos(4x).$$

Problem 5 (**0,5 + 0,5 + 1 = 2 points**)

a) Prove that the following problem is not of Sturm-Liouville type and transform it into one:

$$\begin{cases} x^2\phi'' + x\phi' + \lambda\phi = 0, \\ \phi(1) = 0, \quad \phi(\mathrm{e}) = 0. \end{cases}$$

b) Prove that all the eigenvalues are positive.

c) Find the eigenvalues and eigenfunctions of the problem.

a) A Sturm-Liouville problem is of the form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma\phi = 0.$$

In our case, we should have: $x^2 = p(x), \ x = p'(x)$, that is impossible, so we look for an integrating factor $H(x)$ to transform the problem:

$$\begin{aligned} &H(x)(x^2\phi'' + x\phi' + \lambda\phi) = 0 \quad \Longrightarrow \quad H(x)x^2 = p(x), \quad H(x)x = p'(x) \\ &\Longrightarrow \quad H'(x)x^2 + 2xH(x) = H(x)x \quad \Longrightarrow \quad \frac{H'(x)}{H(x)} = -\frac{1}{x} \quad \Longrightarrow \quad H(x) = \frac{1}{x}. \end{aligned}$$

The problem is now:

$$x\phi''(x) + \phi'(x) + \lambda\frac{1}{x}\phi(x) = \frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \lambda\frac{1}{x}\phi = 0.$$

b) We use the Rayleigh quotient, with $p = x$, $q = 0$, $\sigma = \frac{1}{x}$:

$$\lambda = \frac{\Big[x\phi(x)\phi'(x)\Big]_1^{\mathrm{e}} + \int_1^{\mathrm{e}} x(\phi'(x))^2 dx}{\int_1^{\mathrm{e}} \phi^2(x)\frac{1}{x}\, dx} = \frac{\int_1^{\mathrm{e}} x(\phi'(x))^2 dx}{\int_1^{\mathrm{e}} \phi^2(x)\frac{1}{x}\, dx} \geq 0.$$

Also, $\lambda = 0 \quad \Longrightarrow \quad \phi'(x) = 0 \quad \Longrightarrow \quad \phi(x) = c = \phi(1) = 0$, so $\lambda > 0$.

c) We can solve it as an Euler equation: $\phi = x^r$ and use that $\lambda > 0$:

$$\begin{aligned}
&x^r[r(r-1) + r + \lambda] = 0 \quad \Longrightarrow \quad r^2 = -\lambda \quad \Longrightarrow \quad r = \pm i\sqrt{\lambda} \\
&x^{\pm i\sqrt{\lambda}} = \mathrm{e}^{\pm i(\sqrt{\lambda}\log x)} \quad \Longrightarrow \quad \phi(x) = c_1\cos(\sqrt{\lambda}\log x) + c_2\sin(\sqrt{\lambda}\log x) \\
&\phi(1) = c_1 = 0, \qquad \phi(e) = c_2\sin(\sqrt{\lambda}) = 0.
\end{aligned}$$

We only have a non-zero solution if:

$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, \ldots \qquad \phi_n(x) = \sin(n\pi\log x).$$
