Universidad Carlos III de Madrid
Escuela Politécnica Superior
Departamento de Matemáticas

DIFFERENTIAL EQUATIONS
FINAL EXAM - SOLUTIONS
22th of January, 2018
Degree in Biomedical Engineering.

Time: 3 hours

Problem 1 (**1.5 points**)

Solve the equation
$$\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3.$$

Solution:
This is a Bernoulli equation with $n = 3$, we can make the change:
$$z = y^{1-3} = y^{-2} \quad \Longrightarrow \quad z' = -2y^{-3}y' \quad \Longrightarrow \quad y' = \frac{-z'y^3}{2}.$$

With the change, the new equation is linear, and we solve it:
$$\frac{-z'y^3}{2} - 5y = -\frac{5}{2}xy^3 \quad \Longrightarrow \quad z' + 10z = 5x$$
$$\Longrightarrow \quad z = e^{-\int 10dx}\left[\int 5xe^{\int 10dx}dx + C\right] = e^{-10x}\left[\frac{xe^{10x}}{2} - \frac{e^{10x}}{20} + C\right] = \frac{x}{2} - \frac{1}{20} + Ce^{-10x}$$
$$\Longrightarrow \quad y = \frac{\pm 1}{\sqrt{\frac{x}{2} - \frac{1}{20} + Ce^{-10x}}}.$$

Problem 2 (**2 points**)

Find the general solution of the equation:
$$y'' + y = \mathrm{tg}(x).$$

Solution:
First we solve the homogeneous equation and try $y_h = \mathrm{e}^{rx}$, then:
$$r^2 + 1 = 0 \quad \Longrightarrow \quad r = \pm i \quad \Longrightarrow \quad y_h(x) = c_1\cos(x) + c_2\sin(x),$$

We obtain a particular solution from these two independent solutions, $y_1 = \cos(x)$ and $y_2 = \sin(x)$ using variation of parameters:
$$y_p = v_1y_1 + v_2y_2,$$

where:
$$v_1 = -\int \frac{y_2R(x)}{W(y_1, y_2)}dx, \qquad v_1 = \int \frac{y_1R(x)}{W(y_1, y_2)}dx,$$

$$W(y_1, y_2) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = 1.$$

So:

$$v_1 = -\int \sin(x)\,\mathrm{tg}(x)dx = \int (\cos(x) - \sec(x))dx = \sin(x) - \log|\sec(x) + \mathrm{tg}(x)|.$$
$$v_2 = \int \cos(x)\,\mathrm{tg}(x)dx = \int \sin(x)dx = -\cos(x).$$

Then a particular solution is:

$$y_p = (\sin(x) - \log|\sec(x) + \mathrm{tg}(x)|)\cos(x) - \cos(x)\sin(x) = -\cos(x)\log|\sec(x) + \mathrm{tg}(x)|.$$

The general solution is:

$$y = c_1\cos(x) + c_2\sin(x) - \cos(x)\log|\sec(x) + \mathrm{tg}(x)|.$$

Problem 3 (**1.5 points**)

Solve the integro-differential equation

$$f'(x) + \int_0^x 4 \cdot f(x-t)\,dt = x - \sin x, \qquad f(0) = 2.$$

SOLUTION:
The equation is $f'(x) + 4 * f(x) = x - \sin x$ where $*$ denotes the convolution. Now we apply the Laplace transformation:

$$sF(s) - f(0) + \frac{4}{s} \cdot F(s) = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Then:

$$\frac{s^2+4}{s} \cdot F(s) = \frac{1}{s^2 \cdot (s^2+1)} + 2$$

and:

$$F(s) = \frac{1}{s \cdot (s^2+1^2) \cdot (s^2+2^2)} + \frac{2s}{s^2+2^2}$$

Now we make a decomposition into simple fractions of the first fraction and obtain:

$$F(s) = -\frac{1}{3}\frac{s}{s^2+1^2} + \frac{1}{12} \cdot \frac{s}{s^2+2^2} + \frac{1}{4} \cdot \frac{1}{s} + 2 \cdot \frac{s}{s^2+2^2} = -\frac{1}{3}\frac{s}{s^2+1^2} + \frac{25}{12} \cdot \frac{s}{s^2+2^2} + \frac{1}{4} \cdot \frac{1}{s}$$

Finally, the solution is

$$f(x) = -\frac{1}{3} \cdot \cos x + \frac{25}{12} \cdot \cos 2x + \frac{1}{4}.$$

Problem 4 (**2.5 points**)

Solve, using separation of variables, the telegraph equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2\dfrac{\partial u}{\partial t} + u = \dfrac{\partial^2 u}{\partial x^2}, & t > 0, \quad x \in (0, \pi), \\ u(0,t) = u(\pi,t) = 0, & t > 0, \\ u(x,0) = 4\sin(2x), & x \in [0, \pi], \\ \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in [0, \pi]. \end{cases}$$

Solution:

We consider solutions of the form $u(x,t) = \phi(x)G(t)$ so:

$$G''(t)\phi(x) + 2G'(t)\phi(x) + G(t)\phi(x) = \phi''(x)G(t)$$

$$\frac{G''(t) + 2G(t)}{G(t)} + 1 = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

We also separate the conditions:

$$u(0,t) = 0 \implies \phi(0) = 0, \qquad u(\pi,t) = 0 \implies \phi(\pi) = 0, \qquad u_t(x,0) = 0 \implies G'(0) = 0.$$

The eigenvalue problem is known:

$$\left\{ \begin{array}{l} \phi''(x) + \lambda\phi(x) = 0, \\ \phi(0) = \phi(\pi) = 0. \end{array} \right. \implies \left\{ \begin{array}{l} \phi_n(x) = \sin(nx), \\ \lambda_n = n^2, \end{array} \right. \quad n = 1, 2, \dots$$

The time problem for each $\lambda_n = n^2$ is the following:

$$\left\{ \begin{array}{l} G_n''(t) + 2G_n(t) + (1+n^2)G_n(t) = 0, \\ G_n'(0) = 0. \end{array} \right.$$

With $G_n = \mathrm{e}^{rt}$ we obtain:

$$r^2 + 2r + (1+n^2) = 0 \implies r = \frac{-2 \pm \sqrt{4 - 4(1+n^2)}}{2} = -1 \pm ni.$$

and:

$$G_n(t) = \mathrm{e}^{-t}\Big(c_1\cos(nt) + c_2\sin(nt)\Big).$$

$$G_n'(t) = \mathrm{e}^{-t}\Big(-c_1\cos(nt) - c_2\sin(nt) - c_1 n\sin(nt) + c_2 n\cos(nt)\Big).$$

With the initial condition:

$$G'(0) = 0 = -c_1 + c_2 n \implies c_2 = \frac{c_1}{n}.$$

So

$$G_n(t) = K\mathrm{e}^{-t}\Big(\cos(nt) + \frac{1}{n}\sin(nt)\Big).$$

The product solution, using also the superposition principle is:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\mathrm{e}^{-t}\Big(\cos(nt) + \frac{1}{n}\sin(nt)\Big).$$

With the non homogeneous initial condition we obtain the coefficients:

$$u(x,0) = 4\sin(2x) = \sum_{n=1}^{\infty} B_n\sin(nx) \implies \left\{ \begin{array}{l} B_2 = 4, \\ B_n = 0, \quad n \neq 2. \end{array} \right.$$

The solution is then:

$$u(x,t) = 4\sin(2x)\mathrm{e}^{-t}\Big(\cos(2t) + \frac{1}{2}\sin(2t)\Big).$$

Problem 5 (**0,5 + 0,5 + 1,5 = 2.5 points**)

a) Transform the following into a Sturm-Liouville problem:

$$\begin{cases} r\dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \lambda r^2\phi = 0, \quad 0 < r < 2, \\ \phi(2) = 0, \\ |\phi(0)| < \infty. \end{cases}$$

b) Prove that all the eigenvalues are positive and determine the orthogonality relation satisfied by the eigenfunctions.

c) Find the eigenfunctions and eigenvalues.

SOLUTION:

a) If we divide by r we obtain a Sturm-Liouville equation with $p = r$, $q = 0$ and $\sigma = r$:

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0,$$

This can be done directly or with an integrating factor, that is $H(r) = \dfrac{1}{r}$.

b) We use the Rayleigh quotient to estimate the eigenvalues:

$$\lambda = \frac{\Big[-r\phi(r)\phi'(r)\Big]_0^2 + \displaystyle\int_0^2 r(\phi'(r))^2 dr}{\displaystyle\int_0^2 \phi^2(r) r\, dr} = \frac{\displaystyle\int_0^2 r(\phi'(r))^2 dr}{\displaystyle\int_0^2 \phi^2(r) r\, dr} \geq 0,$$

and

$$\lambda = 0 \quad \Longrightarrow \quad \phi'(r) = 0 \quad \forall r \quad \Longrightarrow \quad \phi(r) = C = \phi(2) = 0.$$

So $\lambda > 0$.

Two eigenfunctions, $\phi_n(r)$ and $\phi_m(r)$ satisfy the orthogonality relation:

$$\int_0^2 \phi_n(r)\phi_m(r) r\, dr = 0 \quad \text{for } n \neq m.$$

c) The original equation is:

$$r^2\phi'' + r\phi' + \lambda r^2\phi = 0$$

Since $\lambda > 0$, we can make the change:

$$z = \sqrt{\lambda} r \quad \Longrightarrow \quad r = \frac{z}{\sqrt{\lambda}}, \quad \frac{d}{dr} = \frac{d}{dz}\frac{dz}{dr} = \sqrt{\lambda}\frac{d}{dz}.$$

The new equation is a Bessel equation of order zero:

$$z^2\phi_z z + z\phi_z + z^2\phi = 0$$

The solution is a linear combination of the first and second kind Bessel functions of order zero:

$$\phi = c_1 J_0(z) + c_2 Y_0(z) = c_1 J_0(r\sqrt{\lambda}) + c_2 Y_0(r\sqrt{\lambda})$$

The boundedness condition implies that $c_2 = 0$, so:

$$\phi(r) = J_0(r\sqrt{\lambda}),$$

and with the boundary condition we obtain the eigenvalues:

$$\phi(2) = 0 = J_0(2\sqrt{\lambda}) \quad \Longrightarrow \quad \lambda_n = \frac{\eta_{0,n}^2}{4}, \quad n = 1, 2, \ldots$$

where the $\eta_{0,n}$ are the infinite zeroes of J_0. The eigenfunctions are:

$$\phi_n(r) = J_0\left(\frac{r\eta_{0,n}}{2}\right), \quad n = 1, 2, \ldots$$
