Module 3: Analysis of AC circuits

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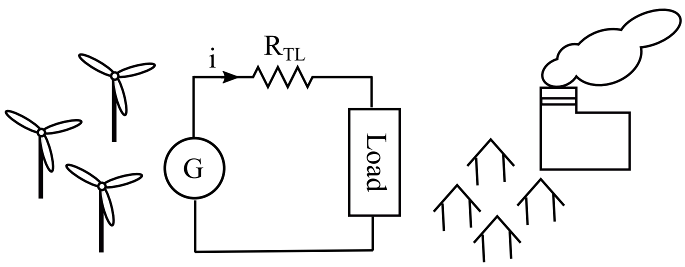
In this module the main concepts of Alternating Current (AC) circuits are introduced. The analysis of circuits in the frequency domain is presented and systematic methods for circuit analysis are applied to the solution of AC circuits. The last part of the module brings in some concepts about the exchange of power in AC circuits.

1 Introduction to AC systems

Most power systems nowadays are based in AC circuits. In these circuits the generators supply voltages and currents that are not constant but vary in time as sinusoidal functions.

The widespread use of AC systems is related with the fact that the transport of energy is more efficient if it is carried out at a higher voltage level. To illustrate this aspect, a case is analysed below in which a certain amount of power p is transferred from a generator to a load. The energy is transported with a transmission line of resistance R_{TL}.



The line losses depend on the current that flows through the line. Then, if the power required by the load is:

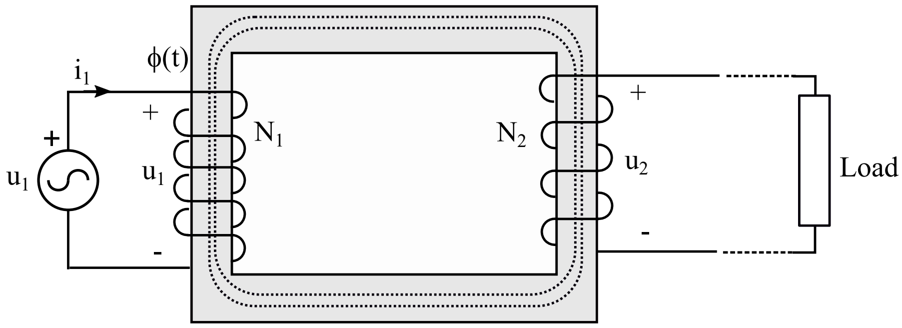
$$p_{load} = u \cdot i \tag{1}$$

the line losses are:

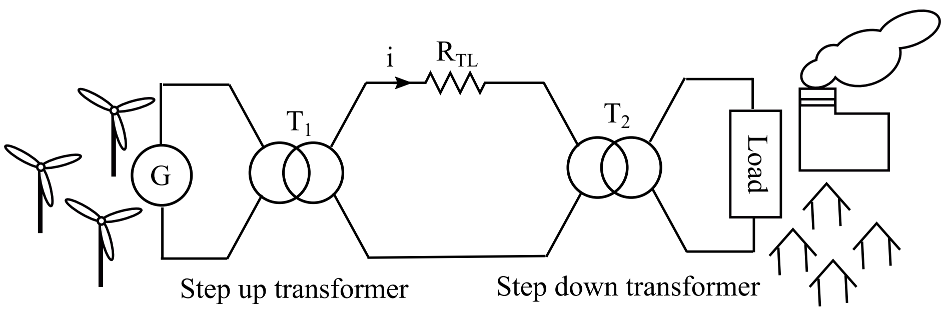
$$p_{loss} = R_{TL} \cdot i^2 \tag{2}$$

If the voltage of the system is raised up, and the energy is transported at a higher voltage level, the current required to transport the same amount of power cuts down by the same rate and and the line losses cuts down by a rate which is the square of the voltage reduction (i.e. if the voltage is multiplied by 10, the current flow to transfer the power p is divided by 10, and the power losses at the lines are divided by 100).

In order to change the voltage level of the energy that is transferred from the generators to the consumers **distribution and power transformers** are used. These machines are constituted by two coupled inductors (the primary and secondary windings) with different numbers of turns. As was studied in Module 1, the self inductance and the mutual inductance effects only take place when the current flowing through the inductors vary with time. This implies that transformers can only be applied in AC systems.



The following diagram shows the typical configuration of a power system. The voltage level of the generated energy is increased with a **step-up transformer** (T_1 in the diagram). The transport of energy is carried out at high voltage, so that the current required to transfer the power demanded by the loads is lower ($p = u \cdot i$) and so, the losses and the voltage drop at the transmission lines are minimized. Later, the voltage level is decreased at the consumption points by means of a **step-down transformer** (T_2 in the diagram). Finally, the energy is supplied to the loads at the required voltage level.



2 Characteristics of a sinusoidal function

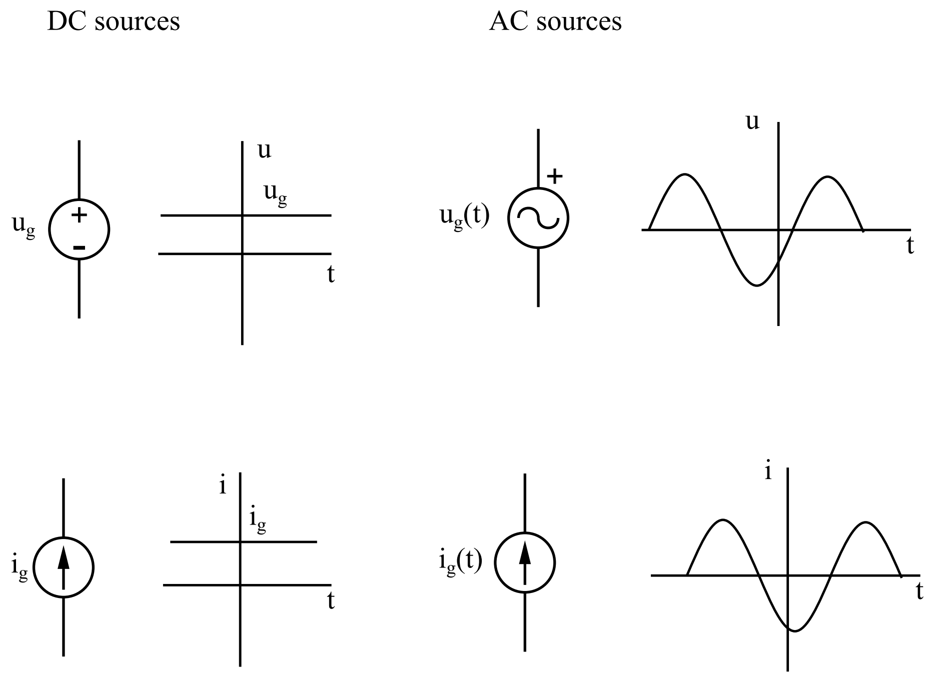
2.1 Sources in AC circuits

In DC circuits voltages and currents do not change in time; the instantaneous voltages and currents are constant:

$$u(t) = u_g \qquad i(t) = i_g \tag{3}$$

In AC circuits the voltages and currents supplied by voltage and current sources are not constant but are defined by sinusoidal functions. This means that the electric variables change over time.

The following diagram shows the typical variation of the voltages and currents in DC and AC voltage and current sources and the symbols that represent these elements:



In AC circuits voltages are currents are defined by sinusoidal functions of the following type:

$$u(t) = U_{max} \cdot \cos(\omega t + \varphi_u) \qquad i(t) = I_{max} \cdot \cos(\omega t + \varphi_i) \tag{4}$$

The sinusoidal functions might also be defined in terms of a function sine, applying the relationship between the sine and cosine functions:

$$sin\alpha = \cos(\alpha - \pi/2) \tag{5}$$

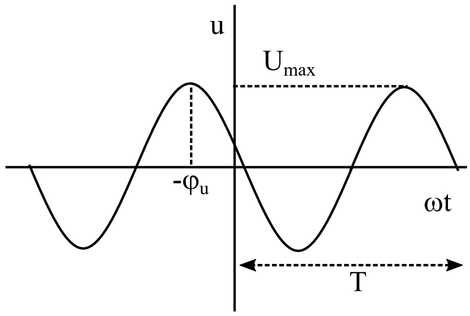
When we analyse an AC circuit all the currents and voltages must be represented either as sine or as cosine functions. Both options are valid but a consistent criteria must be adopted. The same type of function must be considered when comparing or making operations with more than two signals (i.e. when summing or subtracting two voltages or currents). **In this notes we will always use the cosine function to define electric signals**.

2.2 Main parameters of a sinusoidal signal

Typically AC voltages are defined as:

$$u(t) = U_{max} \cdot \cos(\omega t + \varphi_u) \tag{6}$$

It is interesting to plot the previous sinusoidal function and characterize its main parameters.



The main parameters of a sinusoidal function are:

- **Amplitude** (Umax), also referred to as "Peak voltage" and "Maximum voltage". It is the maximum value reached by the function.
- **Period** (T): Time needed to complete a cycle (from one maximum to the next one). It is measured in seconds.
- **Frequency** (f): Number of cycles described in one second. It is measured in Hertz (i.e. cycles per second)

$$f = \frac{1}{T} \qquad [Hz] \tag{7}$$

- **Angular frequency** (ω): Frequency of the function in radians per second.

$$\omega = 2 \cdot \pi \cdot f \qquad [rad] \cdot [s]^{-1} \tag{8}$$

- **Phase angle** (φ): is the phase difference between the maximum of the function and the origin. According to the units of ω and t we see that the phase angle must be expressed in radians. However, for practical reasons, it is very common to express it in degrees. This is not correct from the dimensions prospective but simplifies the analysis in many cases. **In this notes the phase angle will be sometimes expressed in degrees**.
- **Mean value**: The mean value of a sinusoidal function equals zero

$$U_{mean} = \frac{1}{T}\int_{t_0}^{t_0+T} u(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} U_{max} \cdot \cos(\omega t + \varphi_u)dt = 0 \tag{9}$$

- **Root mean square value** (rms) or effective value:

$$U_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} u^2(t)dt} \tag{10}$$

Considering that the voltage varies as a cosine function, we can find a relation between the amplitude and the rms value:

$$U_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} U_{max}^2 \cdot \cos^2(\omega t + \varphi_u)dt} = \frac{U_{max}}{\sqrt{2}} \tag{11}$$

The rms value of the sinusoidal signals is an important parameter and it will be very common to express those functions in terms of rms instead of peak values. To simplify the notation we will denote the rms value of a voltage or current just with the letter u or i in capital letters without any subscripts:

$$U = U_{rms} = \frac{U_{max}}{\sqrt{2}} \qquad I = I_{rms} = \frac{I_{max}}{\sqrt{2}} \tag{12}$$

And we will commonly express the AC voltages and currents as:

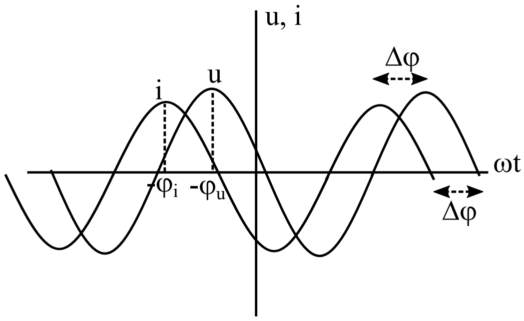
$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i) \tag{13}$$

2.3 Relative phase shift

Another important parameter is the relative **phase shift** between different sinusoidal functions.

The phase shift between two signals is the distance between their zero crossings or their peaks. For example, if the sinusoidal voltage and a sinusoidal current provided as (14) are plotted vs. time, a certain shift between the maximums and zero crossings of both variables is observed:

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i) \tag{14}$$



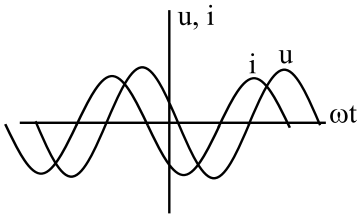
The phase shift can be quantified as:

$$\Delta\varphi_{u,i} = \varphi_u - \varphi_i \tag{15}$$

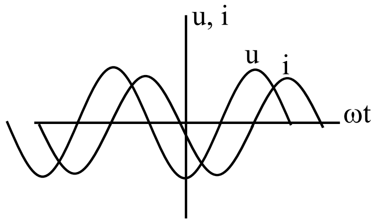
In the example shown in the diagram $\varphi_u < \varphi_i$ and then $\Delta\varphi_{u,i} < 0$. In this case we say that **voltage lags current** and **current leads voltage**.

If, on the contrary, $\Delta\varphi_{u,i}$ was positive the voltage would lead the current.

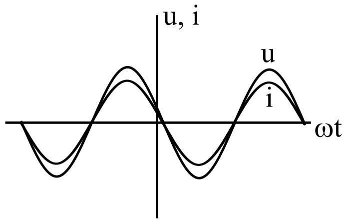
The following diagram summarizes the different situations that can be found regarding the phase shift between two signals:



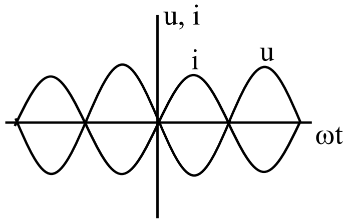
Current leads voltage
Voltage lags current



Voltage leads current
Current lags voltage



Current and voltage are in phase



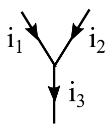
Current and voltage are in antiphase

3 Analysis of AC circuits in the time domain

The following sections tackle the methods that can be used to perform the analysis of AC circuits. An initial approach could consist on the application of the methods that were learnt for the analysis of DC circuits to AC circuits. In this case we would face two main challenges:

1. Operating with sinusoidal functions is not easy. Even doing basic calculations, such as adding two sinusoidal functions, would entail some difficulty.

   Imagine that we want to sum the to currents i_1 and i_2 to find the current i_3 applying Kirchhoff current law:

   $i_1+i_2=i_3$

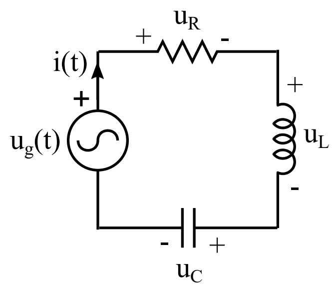
   $$i_1 = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_{i_1}) \quad i_2 = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_{i_2}) \tag{16}$$

   $$i_3 = \sqrt{2} \cdot I_1 \cdot \cos(\omega t + \varphi_{i_1}) + \sqrt{2} \cdot I_2 \cdot \cos(\omega t + \varphi_{i_2}) = \sqrt{2} \cdot I_3 \cdot cos(\omega t + \varphi_{i_3}) \tag{17}$$

   Finding the values I_3 and φ_{i_3} from the parameters of i_1 and i_2 is not immediate and requires complex math analysis.

2. The analysis of AC circuits involves the solution of differential equations or differential systems of equations what complicates the calculations significantly.

   Imagine that we want to analyse the following circuit to find the current i(t) that flows though the three series-connected elements:



The voltage of the source is:

$$u_g(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \tag{18}$$

Then, according to 2KL:

$$u_g(t) = u_R(t) + u_L(t) + u_C(t) \tag{19}$$

The relationships between voltage and current in resistors, inductors and capacitors are:

$$u_R(t) = R \cdot i(t) \qquad u_L(t) = L \cdot \frac{di(t)}{dt} \qquad u_C(t) = \frac{1}{C} \cdot \int i(t)dt \tag{20}$$

Then, replacing (20) in (19):

$$u_g(t) = R \cdot i(t) + L \cdot \frac{di(t)}{dt} + \frac{1}{C} \cdot \int i(t)dt \tag{21}$$

and applying derivatives to (21) we get a differential equation that relates $i(t)$ with u_g, R, L and C:

$$\frac{du_g(t)}{dt} = R \cdot \frac{di(t)}{dt} + L \cdot \frac{d^2 i(t)}{dt^2} + \frac{1}{C} \cdot i(t) \tag{22}$$

To find the current $i(t)$ we have to solve the previous second-order differential equation.

The solution of the differential equation (22) is made up of the sum of two terms: the transient current (i_t), which represents the behaviour of the system after a change (i.e, a connection or disconnection of a source, a change on the voltage level..), and the steady state current (i_{ss}) which represents the behaviour of the system in permanent regime.

$$i(t) = i_t + i_{ss} \tag{23}$$

Taking into account concepts of differential-equations solution, we can state that **if the excitation to the system is a sinusoidal function of frequency ω, the steady-state current will also be a sinusoidal function of the same frequency**.

$$u_g(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \quad => \quad i_{ss}(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i) \tag{24}$$

Since this course is focused in the analysis of the steady state performance of electric circuits, for us, the solution to the problem would be:

$$i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i) \tag{25}$$

Finding the parameters I and φ_i that define the instantaneous current would require solving the differential equation (22).

Some conclusions can be extracted by the previous analysis:

- The analysis of AC circuits involves the solution of differential equations or systems of equations. For simple circuits the resulting equations are not complex but in circuits with more elements (i.e. circuits with several mesh) finding and a solution may be challenging.
- If the excitation of a circuit is a sinusoidal voltage or current of frequency ω, the responses (i.e. all the resulting currents and voltages) are also sinusoidal functions of the same frequency ω.
- Our goal is to find the amplitudes and phase shifts of the responses.

In further sections we will introduce the analysis of AC circuits in the frequency domain. This tool is based in the representation of sinusoidal functions by means of complex numbers and simplifies the analysis of AC circuits greatly. Before introducing the method a short review of complex algebra is provided.

4 Complex numbers: a short review

The irrational number "i" is defined as the square root of -1. In circuit analysis the letter i is used for the variable "current"; to avoid confusion "j" is commonly used to denote the square root of -1:

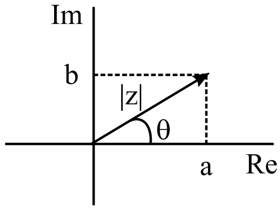
$$j = \sqrt{-1} \tag{26}$$

A complex number can be expressed as the sum of a real part, a, and an imaginary part, b:

$$z = a + bj \tag{27}$$

The former representation of complex numbers is the so-called **rectangular form**.

Sometimes it is useful to represent complex numbers as **vectors in the complex plane**, in which the x axis corresponds to the real parts and the y axis to the imaginary parts:



As can be derived from the previous diagram the complex number z can be also expressed in **polar form**:

$$z = |z|\angle\theta \tag{28}$$

being:

$$|z| = \sqrt{a^2 + b^2} \qquad \theta = \arctan\frac{b}{a} \tag{29}$$

If we want to transform a complex number expressed in polar form into rectangular form we use:

$$a = |z| \cdot \cos\theta \qquad b = |z| \cdot \sin\theta \tag{30}$$

Additionally, complex numbers may be expressed in the so-called **exponential form**:

$$z = |z| \cdot e^{j\theta} \tag{31}$$

Euler's equation provides the relationship between the trigonometric functions and the complex exponential function:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{32}$$

5 Phasor representation of a sinusoidal function

5.1 Definition of phasor

In AC systems the instantaneous values of voltages and currents are defined by cosine functions. Euler's equation (32) provides a relation between sinusoidal and exponential functions; its application allows us to express the instantaneous values of voltages and currents in AC as the real part of an exponential function. Then, an AC voltage of frequency ω, rms value U and phase angle φ_u might be expressed as:

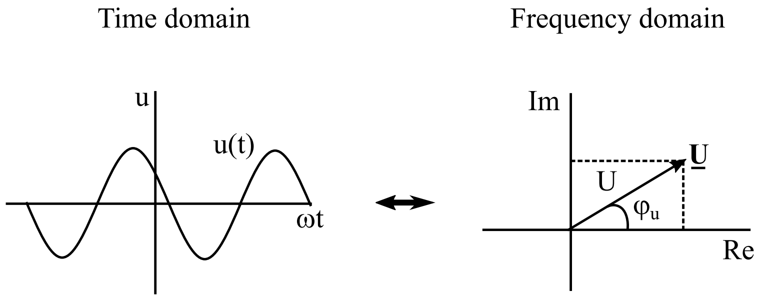
$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) = \sqrt{2} \cdot U \cdot Re(e^{j(\omega t + \varphi_u)}) \tag{33}$$

As the rms value is always a real number, equation (33) may be rearranged as:

$$u(t) = \sqrt{2} \cdot Re(U \cdot e^{j\varphi_u} \cdot e^{j\omega t}) \tag{34}$$

According to the analysis provided in section 3, all the voltages and currents of a circuit excited with AC sources of frequency ω, are sinusoidal functions of the same the frequency ω. The unknown information is the rms value of those signals and their phase angle [1].

The **phasor representation of a sinusoidal signal** is defined as a complex number that contains the information on the rms value and the phase angle of the sinusoidal function. Phasors represents sinusoidal functions in the **frequency domain**:



Instantaneous voltage:

$$u(t) = \sqrt{2} \cdot Re(U \cdot e^{j\varphi_u} \cdot e^{j\omega t})$$

Phasor voltage:

$$\underline{\mathbf{U}} = U \cdot e^{j\varphi_u} = U\angle\varphi_u$$

In these notes the phasors will be written with capital lettters, underlined and in bold text ($\underline{\mathbf{U}}$ $\underline{\mathbf{I}}$). Other references use different notations, as cursive letters, bold text or other symbols ($\mathcal{U}$ $\mathcal{I}$, $\overline{U}$ $\overline{I}$, $\hat{U}$ $\hat{I}$, **U I**...).

The mathematical relationship between the sinusoidal function u(t) and the phasor $\underline{\mathbf{U}}$ is:

$$u(t) = \sqrt{2} \cdot Re(\underline{\mathbf{U}} \cdot e^{j\omega t}) \tag{35}$$

5.2 Kirchhoff's laws in phasor form

One of the important facts about phasor representation is that Kirchoff laws, which are the basis for circuit analysis, are still valid when they are expressed in terms of the phasors that represent the sinusoidal voltages or currents.

5.2.1 Kirchhoff's current law

Kirchhoff's current law (KCL) establishes that the algebraic sum of the currents flowing into a node equals zero:

$$\sum_{k=1}^{n} i_k(t) = 0 \quad => \quad i_1(t) + i_2(t) + ... + i_n(t) = 0 \tag{36}$$

If KCL is applied to an AC circuit, the currents in equation (36) are sinusoidal functions which can be expressed as a function of their phasors:

[1]There is an exception to this statement which is the response of a circuit to sources of different frequency acting simultaneously. That case will be study in a further section

$$i_k(t) = \sqrt{2} \cdot I_k \cdot \cos(\omega t + \varphi_{i,k}) = \sqrt{2} \cdot Re(\underline{\mathbf{I}_k} \cdot e^{j\omega t}) \tag{37}$$

where the phasor $\underline{\mathbf{I}_k}$ is:

$$\underline{\mathbf{I}_k} = I_k \angle \varphi_{i,k} \tag{38}$$

Replacing (37) into (36):

$$\sqrt{2} \cdot Re(\underline{\mathbf{I}_1} \cdot e^{j\omega t}) + \sqrt{2} \cdot Re(\underline{\mathbf{I}_2} \cdot e^{j\omega t}) + ... + \sqrt{2} \cdot Re(\underline{\mathbf{I}_n} \cdot e^{j\omega t}) = 0 \tag{39}$$

simplifying and rearranging:

$$Re(\underline{\mathbf{I}_1} \cdot e^{j\omega t} + \underline{\mathbf{I}_2} \cdot e^{j\omega t} + ... + \underline{\mathbf{I}_n} \cdot e^{j\omega t}) = 0 \tag{40}$$

getting common factor:

$$Re((\underline{\mathbf{I}_1} \cdot + \underline{\mathbf{I}_2} \cdot + ... + \underline{\mathbf{I}_n}) \cdot e^{j\omega t}) = 0 \tag{41}$$

As we know that $e^{j\omega t} \neq 0$, we can state that **the algebraic sum of the phasor representation of the currents that flow into the node equals zero**, what means that KCL is also valid in terms of the phasors current.

$$\underline{\mathbf{I}_1} + \underline{\mathbf{I}_2} + ... + \underline{\mathbf{I}_n} = 0 \qquad => \qquad \sum_k \underline{\mathbf{I}_k} = 0 \tag{42}$$

5.2.2 Kirchhoff's voltage law

Kirchhoff's voltage law (KVL) stablishes that the algebraic sum of the voltages across a closed path of a circuit equals zero:

$$\sum_k u_k = 0 \tag{43}$$

For AC circuits the voltages are defined by sinusoidal functions that can be expressed as a function of their phasor representations:

$$u_k(t) = \sqrt{2} \cdot U_k \cdot \cos(\omega t + \varphi_{u,k}) = \sqrt{2} \cdot Re(\underline{\mathbf{U}_k} \cdot e^{j\omega t}) \tag{44}$$

where:

$$\underline{\mathbf{U}_k} = U_k \angle \varphi_{u,k} \tag{45}$$

Then, if we apply Kirchoff's voltage law to a closed path of a circuit that contains n elements with sinusoidal voltage drops across them, we come to the following equation:

$$\sum_{k=1}^{n} u_k(t) = 0 \quad => \quad u_1(t) + u_2(t) + ... + u_n(t) = 0 \tag{46}$$

Replacing (44) into (46):

$$\sqrt{2} \cdot Re(\underline{\mathbf{U}_1} \cdot e^{j\omega t}) + \sqrt{2} \cdot Re(\underline{\mathbf{U}_2} \cdot e^{j\omega t}) + ... + \sqrt{2} \cdot Re(\underline{\mathbf{U}_n} \cdot e^{j\omega t}) = 0 \tag{47}$$

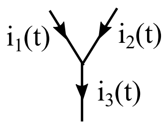
Following the same reasoning that was used to obtain KCL law in phasor form, we find that **the algebraic sum of the phasor representations of the voltages across a closed path of a circuit equals zero**, what means that KVL might be also expressed in terms of the phasors voltage:

$$\underline{\mathbf{U}_1} + \underline{\mathbf{U}_2} + ... + \underline{\mathbf{U}_n} = 0 \quad => \quad \sum_{k} \underline{\mathbf{U}_k} = 0 \tag{48}$$

5.3 Example

Calculate $i_3(t)$ given that:

$$i_1(t) = \sqrt{2} \cdot 10 \cdot \cos(25t + 45^o)A \qquad i_2(t) = \sqrt{2} \cdot 20 \cdot \cos(25t + 90^o)A$$



Solution

Although KCL might be applied to the calculation of the calculation of current i_3, we would need to find the sum of two sinusoidal functions to obtain the amplitude and phase-angle of $i_3(t)$:

$$i_3(t) = i_1(t) + i_2(t) = \sqrt{2} \cdot 10 \cdot \cos(25t + 45^o) + \sqrt{2} \cdot 20 \cdot \cos(25t + 90^o)$$

As an alternative, the calculation might be done in the frequency domain:

1. We obtain the phasors that represent currents $i_1(t)$ and $i_2(t)$:

$$\underline{\mathbf{I}}_1 = 10\angle 45^o A = 7.07 + 7.07jA$$

$$\underline{\mathbf{I}}_2 = 20\angle 90^o A = 20jA$$

2. We apply KCL in phasor form to find phasor $\underline{\mathbf{I}}_3$:

$$\underline{\mathbf{I}}_3 = \underline{\mathbf{I}}_1 + \underline{\mathbf{I}}_2$$

$$\underline{\mathbf{I}}_3 = 10\angle 45^o + 20\angle 90^o = 7.07 + 27.07j = 28\angle 75.36^o A$$

3. Finally, we go back to the time domain and calculate $i_3(t)$

$$i_3(t) = \sqrt{2} \cdot 28 \cdot \cos(25t + 75.36^o) A$$

6 Complex impedance

6.1 Definition

As was explained before, the use of phasors allows us operating with sinusoidal functions in an effective and simple way.

The other difficulty involved in the analysis of AC circuits is the fact that, for capacitors and inductors, the relationships between voltages and currents are given by differential equations, which are more difficult to handle than the linear equations that relate voltages and currents in resistors.

$$u_L(t) = L \cdot \frac{di_L(t)}{dt} \qquad i_C(t) = C \cdot \frac{du_C(t)}{dt} \qquad u_R(t) = R \cdot i_R(t) \tag{49}$$

To avoid the need of solving differential equations in the analysis of AC circuits, a new variable is introduced that relates the phasors voltage and current of passive elements with a linear relation.

The **impedance** (Z) of a passive element is defined as the ratio between the phasor voltage and the phasor current at this element.

$$Z = \frac{\underline{\mathbf{U}}}{\underline{\mathbf{I}}} \tag{50}$$

The unit for the impedance in the SI is Ohm ($[V] \cdot [A]^{-1} = [\Omega]$). This is also true for impedances associated to inductors and capacitors, despite L and C are measured in Henry and Farads .

Additionally we define the **admitance** (Y) as the inverse of the impedance. For a passive element of impedance Z the admitance is:

$$Y = \frac{1}{Z} \tag{51}$$

The admitance is measured in Siemens ($[S] = [\Omega]^{-1}$) in the SI.

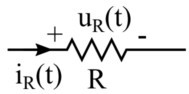
By representing the three types of passive elements as impedances we are able to establish a linear relation between the phasors currents and voltage which is also valid for inductors and capacitors. This relation is the so called **Ohm's law in the frequency domain**:

$$\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}} \tag{52}$$

In the following subsections the expressions to calculate the complex impedance for resistors, inductors and capacitors are obtained.

6.2 Impedance of a resistor

We want to find an expression to calculate the impedance of resistors. To this end, we consider a resistor R with a sinusoidal current flowing through it $i_R(t)$ which causes a voltage drop $u_R(t)$ which is sinusoidal too.



Given that the expression for the current is:

$$i_R(t) = \sqrt{2} \cdot I_R \cdot \cos(\omega t + \varphi_i) \tag{53}$$

The voltage across the resistor calculated with Ohm's law would be:

$$u_R(t) = R \cdot i_R(t) \tag{54}$$

$$u_R(t) = \sqrt{2} \cdot R \cdot I_R \cdot \cos(\omega t + \varphi_i) \tag{55}$$

Moving to the frequency domain, we find that the phasors current and voltage are:

$$\underline{\mathbf{I}}_R = I_R \angle \varphi_i \qquad \underline{\mathbf{U}}_R = R \cdot I_R \angle \varphi_i \tag{56}$$

Then, the impedance of a resistor is:

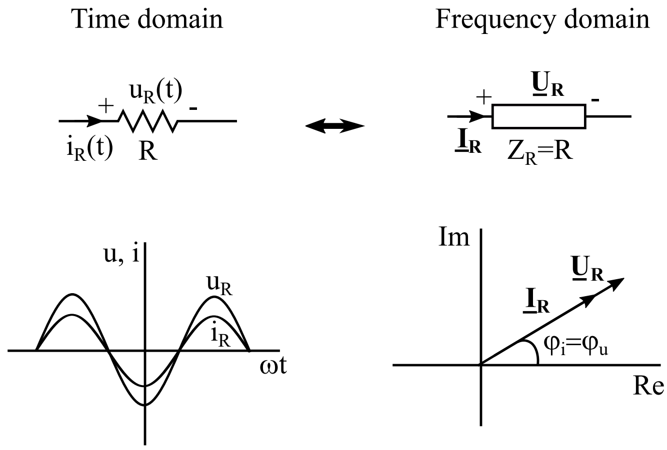
$$Z_R = \frac{\underline{\mathbf{U}}_R}{\underline{\mathbf{I}}_R} = \frac{R \cdot I_R \angle \varphi_i}{I_R \angle \varphi_i} = R \tag{57}$$

In the frequency domain resistors will be represented by means an impedance $Z_R = R$ so that Ohm's law in the frequency domain is verified

$$\underline{\mathbf{U}}_R = Z_R \cdot \underline{\mathbf{I}}_R \tag{58}$$

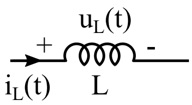
It is interesting to look at the phase shift between the current and the voltage across the resistor. As can be seen both magnitudes are in-phase.

$$\varphi_i = \varphi_u \qquad \Delta\varphi_{u,i} = 0 \tag{59}$$



6.3 Impedance of an inductor

To obtain an expression to calculate the impedance of inductors, we consider the case of an inductor of self inductance L with a sinusoidal current $i_L(t)$ flowing through it.



Considering that the current is:

$$i_L(t) = \sqrt{2} \cdot I_L \cdot \cos(\omega t + \varphi_i) \tag{60}$$

The voltage across it can be calculated taking into account the relationship between the voltage and current in an inductor:

$$u(t) = L \cdot \frac{di_L(t)}{dt} \tag{61}$$

Then the voltage $u(t)$ is:

$$u_L(t) = -\sqrt{2} \cdot \omega \cdot L \cdot I_L \cdot sin(\omega t + \varphi_i) = -\sqrt{2} \cdot \omega \cdot L \cdot I_L \cdot \cos(\omega t + \varphi_i - \frac{\pi}{2}) \tag{62}$$

Moving to the frequency domain we find that the phasors current and voltage are:

$$\underline{\mathbf{I}}_L = I_L \angle \varphi_i \tag{63}$$

$$\underline{\mathbf{U}}_L = -\omega \cdot L \cdot I_L \angle \varphi_i - \frac{\pi}{2} \tag{64}$$

Taking into account that the modulus of a complex number in polar form must always be positive, we write the factor -1 as $1\angle\pi$, and operate in the expression of the phasor voltage:

$$\underline{\mathbf{U}}_L = (1\angle\pi)\cdot(\omega\cdot L\cdot I_L\angle\varphi_i - \frac{\pi}{2}) = \omega\cdot L\cdot I_L\angle\varphi_i + \frac{\pi}{2} \tag{65}$$

Then, the impedance of the inductor is:

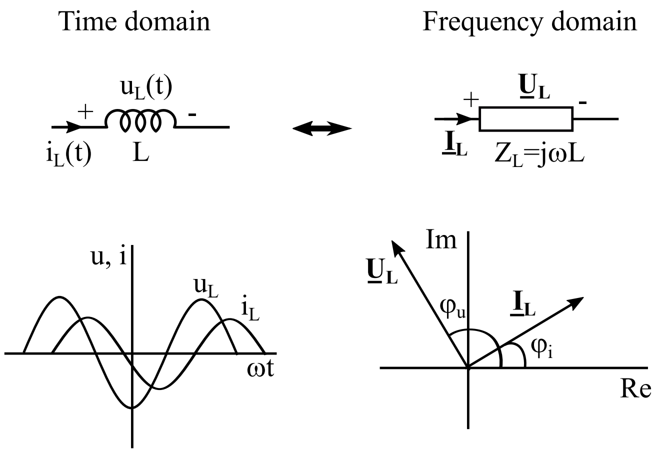
$$Z_L = \frac{\underline{\mathbf{U}}_L}{\underline{\mathbf{I}}_L} = \frac{\omega\cdot L\cdot I_L\angle\varphi_i + \frac{\pi}{2}}{I_L\angle\varphi_i} = \omega\cdot L\angle\frac{\pi}{2} = j\omega L \tag{66}$$

In the frequency domain inductors are represented by means an impedance $Z_L = j\omega L$ and Ohm's law in the frequency domain is verified:

$$\underline{\mathbf{U}}_L = Z_L\cdot\underline{\mathbf{I}}_L \tag{67}$$

If we calculate the phase shift between the current and the voltage across the inductor, we find that the voltage leads the current by 90^o.

$$\varphi_u = \varphi_i + \frac{\pi}{2} \qquad \Delta\varphi_{u,i} = \frac{\pi}{2} = 90^o \tag{68}$$



6.4 Impedance of an mutual inductance

If we want to analyse the voltages and currents of coupled inductors, we should represent the mutual inductance as an impedance too. The expression for the impedance of a mutual inductance M can be obtained following an analogous reasoning as to find the impedance of an inductor:

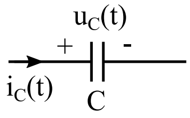
$$Z_M = j\omega M \tag{69}$$

6.5 Impedance of a capacitor

The relation between voltage and current in a capacitor is

$$i(t) = C\cdot\frac{du(t)}{dt} \tag{70}$$

To find an expression to calculate the impedance of capacitors, we obtain the current that flows trough a capacitor of capacitance C if a certain voltage drop $u_C(t)$ is applied across its terminals.



If the voltage across the capacitor is:

$$u_C(t) = \sqrt{2} \cdot U_C \cdot \cos(\omega t + \varphi_u) \tag{71}$$

The current is:

$$i(t) = C \cdot \frac{du_C(t)}{dt} = -\sqrt{2} \cdot \omega \cdot C \cdot U_C \cdot sin(\omega t + \varphi_u) \tag{72}$$

Expressing the negative sign as a factor $1\angle\pi$ and transforming the sine function into a cosine function:

$$i(t) = \sqrt{2} \cdot \omega \cdot C \cdot U_C \cdot \cos(\omega t + \varphi_u - \frac{\pi}{2} + \pi) \tag{73}$$

Moving to the frequency domain, we find that the phasors voltage and current are:

$$\underline{\mathbf{U}}_C = U_C\angle\varphi_u \tag{74}$$

$$\underline{\mathbf{I}}_C = j\omega \cdot C \cdot U_C\angle\varphi_u \tag{75}$$

Then, the impedance of the capacitor is:

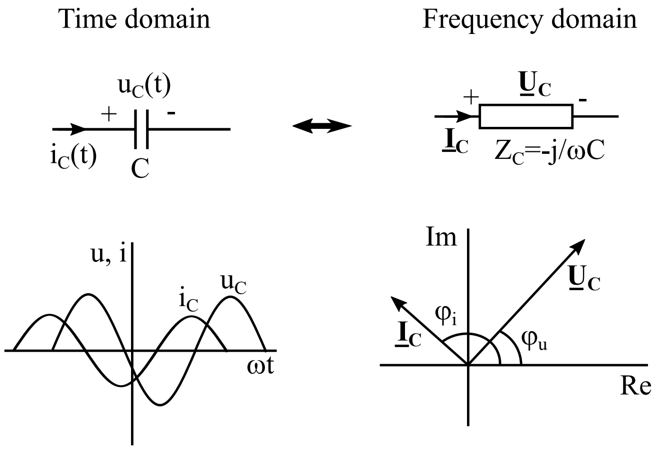
$$Z_C = \frac{\underline{\mathbf{U}}_C}{\underline{\mathbf{I}}_C} = \frac{U_C\angle\varphi_u}{j\omega \cdot C \cdot U_C\angle\varphi_u} = \frac{1}{j\omega \cdot C} = \frac{-j}{\omega \cdot C} \tag{76}$$

In the frequency domain capacitors are represented with an impedance $Z_C = -j/\omega C$ and Ohm's law in the frequency domain is verified:

$$\underline{\mathbf{U}}_C = Z_C \cdot \underline{\mathbf{I}}_C \tag{77}$$

In this case, the phase shift between the current and the voltage across the capacitor is also 90^o, but now the current leads the voltage by 90^o.

$$\varphi_i = \varphi_u + \frac{\pi}{2} \qquad \Delta\varphi_{u,i} = -\frac{\pi}{2} = -90^o \tag{78}$$



6.6 Association of impedances

The representation of resistors, inductors and capacitors by means of impedances opens the possibility of associating passive elements of different nature to obtain an equivalent impedance. The association is possible because in the frequency domain the relation between $\underline{\mathbf{U}}$ and $\underline{\mathbf{I}}$ for the three types of passive elements is the same:

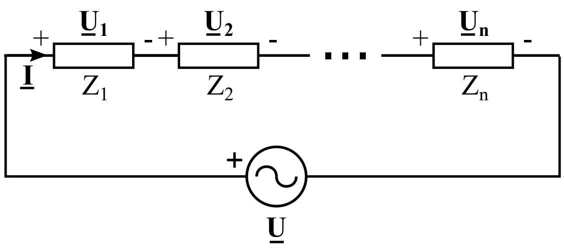
$$\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}} \tag{79}$$

The association of passive elements of different nature (R, L, C) is not possible in the time domain, because each type of element verifies a different relation between u(t) and i(t):

$$u_R(t) = R \cdot i_R(t) \qquad u_L(t) = L \cdot \frac{di_L(t)}{dt} \qquad i_C(t) = C \cdot \frac{du_C(t)}{dt} \tag{80}$$

6.6.1 Series equivalent impedance

Two or more impedances are series connected if the same current flows through them. The n impedances in the figure are series connected, since the current $\underline{\mathbf{I}}$ that goes through all of them is the same.



Each impedance has a phasor voltage drop that is given by Ohm's law in phasor form:

$$\begin{gathered} \underline{\mathbf{U}}_1 = Z_1 \cdot \underline{\mathbf{I}}_1 \\ \underline{\mathbf{U}}_2 = Z_2 \cdot \underline{\mathbf{I}}_2 \\ \dots \\ \underline{\mathbf{U}}_n = Z_n \cdot \underline{\mathbf{I}}_n \end{gathered}$$

The total phasor voltage $\underline{\mathbf{U}}$ can be calculated using 2KL in phasor form:

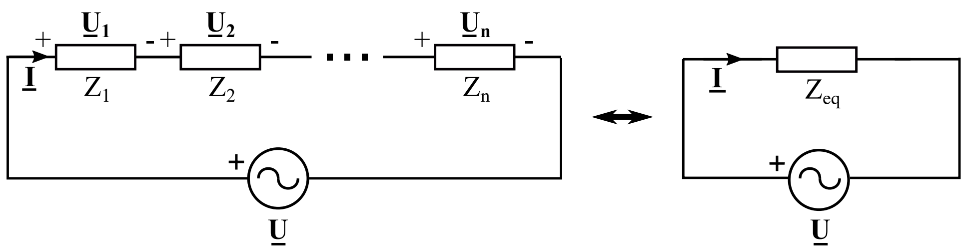
$$\underline{\mathbf{U}} = \underline{\mathbf{U}}_1 + \underline{\mathbf{U}}_2 + ... + \underline{\mathbf{U}}_n = Z_1 \cdot \underline{\mathbf{I}} + Z_2 \cdot \underline{\mathbf{I}} + ... + Z_n \cdot \underline{\mathbf{I}} = (Z_1 + Z_2 + ... + Z_n) \cdot \underline{\mathbf{I}} \quad (81)$$

The set of n impedances can be redrawn as an equivalent impedance Z_{eq}

$$Z_{eq} = Z_1 + Z_2 + ... + Z_n = \sum_{k=1}^{n} Z_k \quad (82)$$

Then,

$$\underline{\mathbf{U}} = Z_{eq} \cdot \underline{\mathbf{I}} \quad (83)$$



Seen from the voltage source the effect of the n impedances connected in series is identical to the effect of the equivalent impedance and it would not be possible to distinguish both configurations. Then we say that both configurations are **equivalent**.

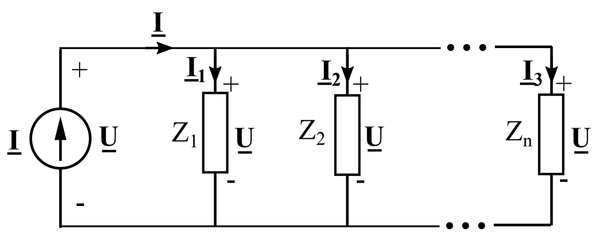
The voltage divider equation

The voltage divider equation is also valid in the frequency domain. If we have n impedances series-connected, and the phasor voltage across the whole set is $\underline{\mathbf{U}}$ the phasor voltage across the impedance k can be calculated with the equation below:

$$\underline{\mathbf{U}}_k = \frac{Z_k}{Z_{eq}} \cdot \underline{\mathbf{U}} \quad (84)$$

6.6.2 Parallel equivalent impedance

Two or more impedances are parallel connected if they have the same phasor voltage across them. The n impedances in the figure are in parallel since the phasor voltage $\underline{\mathbf{U}}$ across all of them is the same.



Each impedance has a different current flow:

$$\underline{\mathbf{I}}_1 = \frac{\underline{\mathbf{U}}}{Z_1} = \underline{\mathbf{U}} \cdot Y_1$$

$$\underline{\mathbf{I}}_2 = \frac{\underline{\mathbf{U}}}{Z_2} = \underline{\mathbf{U}} \cdot Y_2$$

$$\underline{\mathbf{I}}_n = \frac{\underline{\mathbf{U}}}{Z_n} = \underline{\mathbf{U}} \cdot Y_n$$

The total phasor current $\underline{\mathbf{I}}$ can be calculated according to KCL in phasor form:

$$\underline{\mathbf{I}} = \underline{\mathbf{I}}_1 + \underline{\mathbf{I}}_2 + ... + \underline{\mathbf{I}}_n = \frac{\underline{\mathbf{U}}}{Z_1} + \frac{\underline{\mathbf{U}}}{Z_2} + ... + \frac{\underline{\mathbf{U}}}{Z_n} = \underline{\mathbf{U}} \cdot (\frac{1}{Z_1} + \frac{1}{Z_2} + ... + \frac{1}{Z_n}) \tag{85}$$

The set of n impedances can be redrawn as an equivalent impedance Z_{eq}

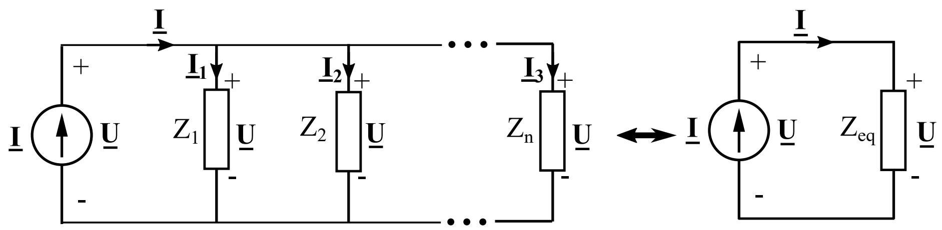
$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + ... + \frac{1}{Z_n} = \sum_{k=1}^{n} \frac{1}{Z_k} \tag{86}$$

$$\underline{\mathbf{I}} = \frac{\underline{\mathbf{U}}}{Z_{eq}} \tag{87}$$

The equations could also be expressed in terms of admitance:

$$Y_{eq} = \sum_{k=1}^{n} Y_k \tag{88}$$

$$\underline{\mathbf{I}} = \underline{\mathbf{U}} \cdot Y_{eq} \tag{89}$$



Seen from the current source, the effect of the n impedances connected in parallel is identical to the effect of the equivalent impedance. We say that both configurations are **equivalent**.

The current divider equation

The current divider equation can also be used in the frequency domain. Given a set of n impedances parallel connected, which are supplied with a total current $\underline{\mathbf{I}}$, the current flowing through the impedance k is:

$$\underline{\mathbf{I}}_k = \frac{Y_k}{Y_{eq}} \cdot \underline{\mathbf{I}} \tag{90}$$

In the particular case of two impedances connected in parallel the expressions for the currents $\underline{\mathbf{I}}_1$ and $\underline{\mathbf{I}}_2$ are:

$$\underline{\mathbf{I}}_1 = \frac{Z_2}{Z_1 + Z_2} \cdot \underline{\mathbf{I}} \tag{91}$$

$$\underline{\mathbf{I}}_2 = \frac{Z_1}{Z_1 + Z_2} \cdot \underline{\mathbf{I}} \tag{92}$$

6.7 Components of a complex impedance

We have studied that the impedances that represent the three passive elements are:

$$Z_R = R \in \mathbb{R} \qquad Z_L = j\omega \cdot L \in \mathbb{C} \qquad Z_C = \frac{-j}{\omega \cdot C} \in \mathbb{C} \tag{93}$$

If we add two or more elements of different nature to obtain the series or parallel equivalent impedances in some cases we will find impedances with real and imaginary part:

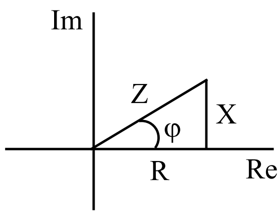
$$Z = R + jX \tag{94}$$

The real part of the impedance always comes from resistive elements, and it is called the **Resistance** (R). The imaginary part always comes from inductors and capacitors (the so called "reactive elements") and is called the **Reactance** (X).

The following table summarizes the values of the impedance, resistance and reactance for resistors, inductors and capacitors.

	Resistance	**Reactance**	**Impedance**
Resistor	R	0	R
Inductor	0	$\omega \cdot L$	$j\omega \cdot L$
Capacitor	0	$-1/\omega \cdot C$	$-j/\omega \cdot C$

Sometimes the complex impedance is plotted as an **Impedance triangle**, which is the representation of the impedance in the complex plane.



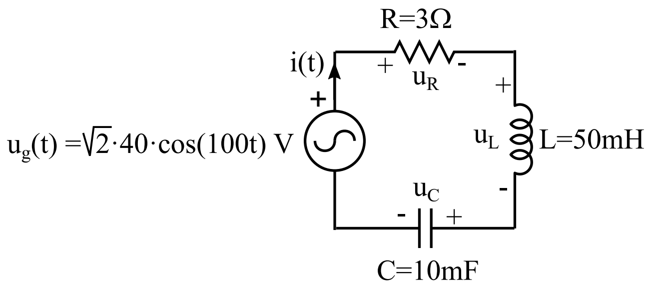
The angle of the complex impedance can be calculated as:

$$\varphi = \arctan \frac{X}{R} \tag{95}$$

The cosine of the angle φ is the so called **power factor** of the impedance. The physical meaning and the practical relevance of the power factor will be studied in further sections.

6.8 Example

In the following circuit the resistor, inductor and capacitor are series-connected and fed with an AC voltage source.



Calculate the current i(t) flowing through the circuit and the voltage drop across each element $u_R(t)$, $u_L(t)$, $u_C(t)$. Draw a phasor diagram with all the voltages and currents.

Solution

The frequency of the source is ω=100 rad/s. First, we obtain the circuit in the frequency domain:

1. Calculation of phasors:

$$\underline{\mathbf{U}}_g = 40\angle 0^o V$$

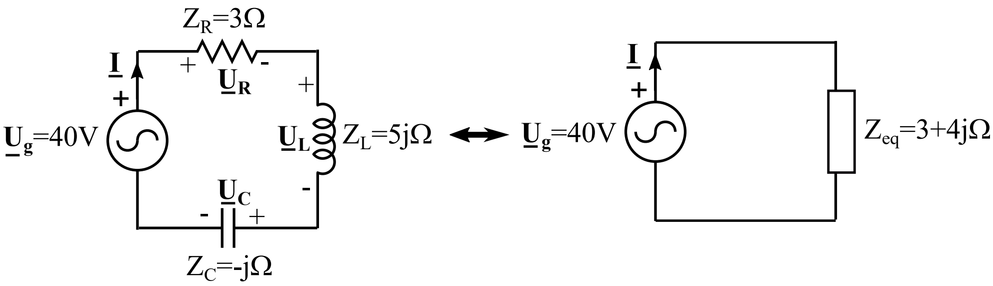
2. Calculation of impedances:

$$Z_R = R = 3\Omega$$

$$Z_L = j\omega \cdot L = j0.05 \cdot 100 = 5j\Omega$$

$$Z_C = \frac{-j}{\omega \cdot C} = \frac{-j}{100 \cdot 0.01} = -j\Omega$$

3. We solve the circuit in the frequency domain:



As the three impedances are series-connected we can calculate the phasor current as:

$$\underline{\mathbf{I}} = \frac{\underline{\mathbf{U}}_g}{Z_R + Z_L + Z_C} = \frac{40}{3 + 5j - j} = 8\angle - 53.13^o A$$

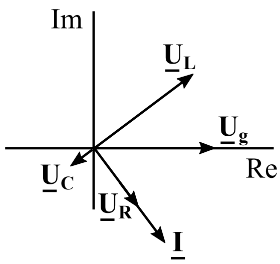
The phasors voltage across the three elements are calculated applying Ohm's law:

$$\underline{\mathbf{U}}_R = \underline{\mathbf{I}} \cdot Z_R = 3 \cdot 8\angle - 53.13^o = 24\angle - 53.13^o V$$

$$\underline{\mathbf{U}}_L = \underline{\mathbf{I}} \cdot Z_L = 5j \cdot 8\angle - 53.13^o = 40\angle 36.87^o V$$

$$\underline{\mathbf{U}}_C = \underline{\mathbf{I}} \cdot Z_C = -j \cdot 8\angle - 53.13^o = 8\angle - 143.13^o V$$

Phasor diagram:



4. Finally, we obtain the variables in the time domain using the phasors obtained in the frequency domain:

$$i(t) = \sqrt{2} \cdot 8 \cdot \cos(100 \cdot t - 53.13^o)A$$

$$u_R(t) = \sqrt{2} \cdot 24 \cdot \cos(100 \cdot t - 53.13^o)V$$

$$u_L(t) = \sqrt{2} \cdot 40 \cdot \cos(100 \cdot t + 36.87^o)V$$

$$u_C(t) = \sqrt{2} \cdot 8 \cdot \cos(100 \cdot t - 143.13^o)V$$

7 Analysis of AC circuits

7.1 General remarks

The representation of circuits in the frequency domain makes possible the application of all the circuit-solving methods that were studied for the analysis of DC circuits to the analysis of AC circuits. Some general remarks are given next:

- To solve AC circuits in the frequency domain we represent all the sinusoidal currents and voltages by phasors and the resistors, inductors and capacitors by impedances:

$$\underline{\mathbf{U}} = U_{rms}\angle\varphi_u \qquad \underline{\mathbf{I}} = I_{rms}\angle\varphi_i \tag{96}$$

$$Z_R = R \qquad Z_L = j\omega \cdot L \qquad Z_C = \frac{-j}{\omega \cdot C} \tag{97}$$

- Kirchhoff's laws are applied in phasor form:

$$\sum_{node} \underline{\mathbf{I}} = 0 \qquad \sum_{mesh} \underline{\mathbf{U}} = 0 \tag{98}$$

- Ohm's law is verified for all passive elements:

$$\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}} \tag{99}$$

- Mesh current analysis and node voltage analysis methods can be applied to the analysis of AC circuits in the frequency domain. Thevenin's theorem is also fulfilled.
- After solving a circuit in the frequency domain we will move back to the time domain and provide the instantaneous currents an voltages as sinusoidal functions.

7.2 Mesh current method

7.2.1 Application of the method

The application of mesh current method in AC circuits is analogous to the method that was learnt for DC circuits. The next steps must be followed:

1. Assign a **phasor mesh current** to each mesh of the circuit
2. Apply KVL in phasor form to every mesh of the circuit applying a consistent sign criteria (in this notes we consider voltage drops as positive and voltage rises as negative) and find a system of equations with the mesh currents as unknowns.
3. Solve the equations to find the currents. In the case of AC circuits it is particularly useful to express the systems of equations in matrix form.

   Mesh equations in matrix form are:

$$[Z] \cdot [\underline{\mathbf{I}}_{mesh}] = [\underline{\mathbf{U}}_g] \tag{100}$$

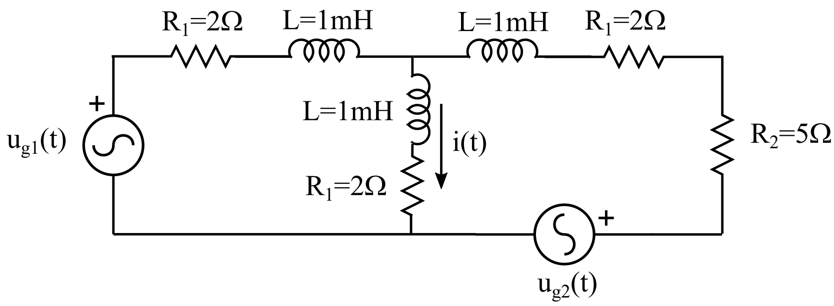
   Now $[Z]$ is the **impedance matrix** whose terms are:

Zii=Sum of the impedances in mesh i

Zij= - Sum of the impedances shared by mesh i and j

7.2.2 Example

Given the following circuit



where:

$$u_{g1}(t) = \sqrt{2} \cdot 50 \cdot \cos(1000 \cdot t)V$$

$$u_{g2}(t) = \sqrt{2} \cdot 30 \cdot \cos(1000 \cdot t + 90^o)V$$

Calculate i(t) using mesh current analysis.

Solution

We find the circuit in the frequency domain (ω=1000 rad/s)

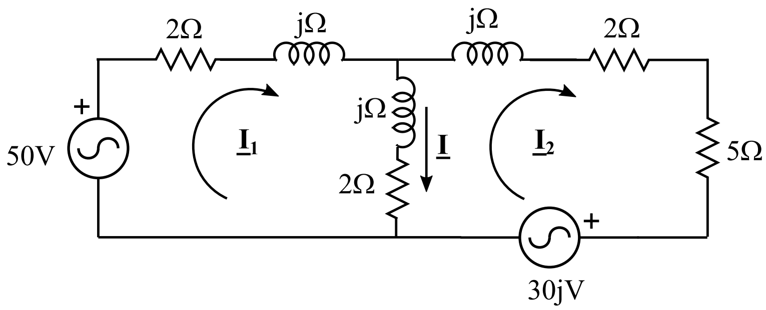
1. Impedances:

$$Z_L = j\omega \cdot L = j \cdot 1000 \cdot 0.001 = j\Omega \qquad Z_{R1} = 2\Omega \qquad Z_{R2} = 5\Omega$$

2. Phasors:

$$\underline{\mathbf{U}}_{g1} = 50\angle 0^o V = 50V$$

$$\underline{\mathbf{U}}_{g2} = 30\angle 90^o V = 30jV$$



We apply 2KL to find the mesh equations of the system:

Mesh 1: $-50 + (2+j) \cdot \underline{\mathbf{I}}_1 + (2+j) \cdot (\underline{\mathbf{I}}_1 - \underline{\mathbf{I}}_2) = 0$

Mesh 2: $30j + (j+2+5) \cdot \underline{\mathbf{I}}_2 + (2+j)(\underline{\mathbf{I}}_2 - \underline{\mathbf{I}}_1) = 0$

We write the equations in matrix form:

$$\begin{pmatrix} 2+j+j+2 & -j-2 \\ -j-2 & j+2+5+2+j \end{pmatrix} \cdot \begin{pmatrix} \underline{\mathbf{I}}_1 \\ \underline{\mathbf{I}}_2 \end{pmatrix} = \begin{pmatrix} 50 \\ -30j \end{pmatrix}$$

$$\begin{pmatrix} 4+2j & -j-2 \\ -j-2 & 2j+9 \end{pmatrix} \cdot \begin{pmatrix} \underline{\mathbf{I}}_1 \\ \underline{\mathbf{I}}_2 \end{pmatrix} = \begin{pmatrix} 50 \\ -30j \end{pmatrix}$$

Solving the equations we find that the phasors mesh current are:

$$\underline{\mathbf{I}}_1 = 11.17 - 7.09j = 13.23\angle - 32.42^o A$$

$$\underline{\mathbf{I}}_2 = 2.34 - 4.19j = 4.8\angle - 60.8^o A$$

Then the phasor $\underline{\mathbf{I}}$ is:

$$\underline{\mathbf{I}} = \underline{\mathbf{I}}_1 - \underline{\mathbf{I}}_2 = 8.83 - 2.9j = 9.3\angle - 18.21^o A$$

And the instantaneous current i(t) is:

$$i(t) = \sqrt{2} \cdot 9.3 \cdot \cos(1000 \cdot t - 18.21^o) A$$

7.3 Node voltage method

7.3.1 Application of the method

The application of node voltage method to the solution of AC circuits is analogous to the method that we learnt for DC circuits. The next steps can be followed to solve any circuit using the method:

1. Assign a **phasor node voltage** to each node of the circuit and label one of them as **reference node**. We consider that the reference node is connected to ground ($\underline{\mathbf{U}}_{rn} = 0V$)
2. Apply KCL in phasor form to each node of the circuit with a consistent sign criteria (in this notes we consider currents flowing out of a node as positive and currents flowing into a node as negative) and find a system of equations with the node voltages as unknowns.
3. Solve the equations to find the node voltages. The equations can be expressed in matrix form to make the calculations easier.

$$[Y] \cdot [\underline{\mathbf{U}}_{node}] = [\underline{\mathbf{I}}_g] \tag{101}$$

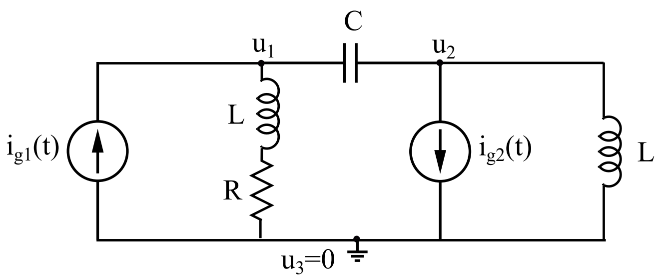
   now $[Y]$ is the **Admitance Matrix** whose terms are:

   Yii=Sum of the admitances connected to node i

   Yij= - Sum of the admitances shared by nodes i and j

7.3.2 Example

Solve the following circuit using nodal analysis and find the instantaneous voltages $u_1(t)$ and $u_2(t)$.



$$R = 2\Omega \qquad C = 1/5F \qquad L = 0.05H$$

$$i_{g1}(t) = \sqrt{2} \cdot 3 \cdot \cos(10 \cdot t)A$$

$$i_{g2}(t) = \sqrt{2} \cdot \cos(10 \cdot t + 90^o)A$$

Solution

We obtain the circuit in the frequency domain ($\omega = 10 rad/s$)

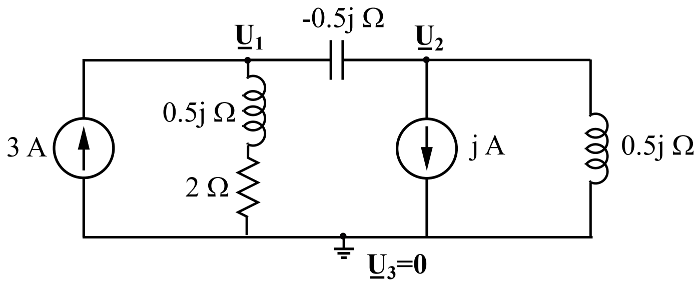
1. Impedances:

$$Z_R = R = 2\Omega \quad Z_L = j \cdot \omega \cdot L = 0,5j\Omega \quad Z_C = -j/\omega \cdot C = -0,5j\Omega$$

2. Phasors:

$$\underline{\mathbf{I}}_{g1} = 3\angle 0^o A = 3A$$

$$\underline{\mathbf{I}}_{g2} = 1\angle 90^o A = jA$$



Nodal equations:

$$\text{Equation for node 1:} \quad -3 + \frac{\underline{\mathbf{U}}_1}{2+0.5j} + \frac{\underline{\mathbf{U}}_1 - \underline{\mathbf{U}}_2}{-0.5j} = 0$$

$$\text{Equation for node 2:} \quad j + \frac{\underline{\mathbf{U}}_2}{0.5j} + \frac{\underline{\mathbf{U}}_2 - \underline{\mathbf{U}}_1}{-0.5j} = 0$$

The nodal equations in matrix form would be:

$$\begin{pmatrix} \frac{1}{2+0.5j} - \frac{1}{0.5j} & -\frac{1}{-0.5j} \\ -\frac{1}{-0.5j} & \frac{1}{0.5j} - \frac{1}{0.5j} \end{pmatrix} \cdot \begin{pmatrix} \underline{\mathbf{U}}_1 \\ \underline{\mathbf{U}}_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -j \end{pmatrix}$$

Solving the equations:

$$\underline{\mathbf{U}}_1 = 0.5V$$

$$\underline{\mathbf{U}}_2 = 0.47 + 1.38j = 1.46\angle 71.2^o V$$

The voltages in the time domain are:

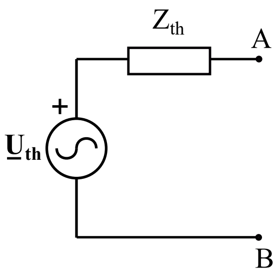
$$u_1(t) = \sqrt{2} \cdot 0.50 \cdot \cos(10 \cdot t)V$$

$$u_2(t) = \sqrt{2} \cdot 1.46 \cdot \cos(10 \cdot t + 71.2^o)V$$

7.4 Thevenin equivalent

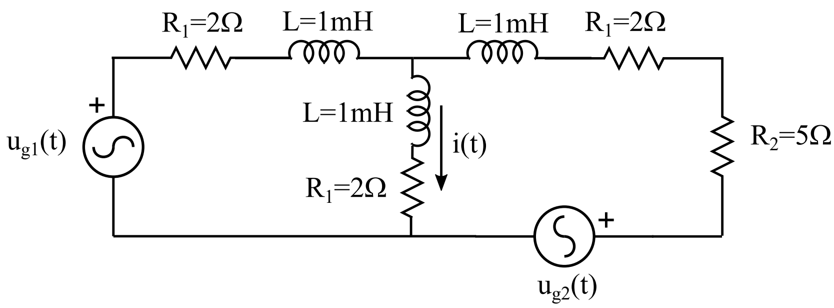
Thenvenin's theorem is valid for AC circuits in the frequency domain. Thevenin equivalent of a circuit in the frequency domain consists of a voltage source of value $\underline{\mathbf{U}}_{th}$ and a series impedance Z_{th}.

The methods that can be followed to obtain the parameters of the equivalent are analogous to the ones studied for DC circuits.



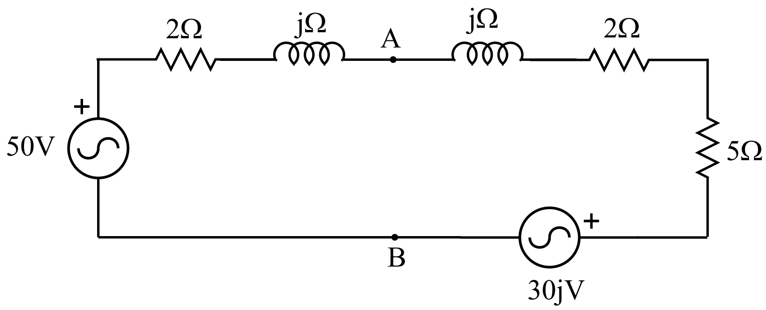
7.4.1 Example 1

In the circuit of the example of section 7.2.2, calculate the current i(t) using Thevenin's Theorem.

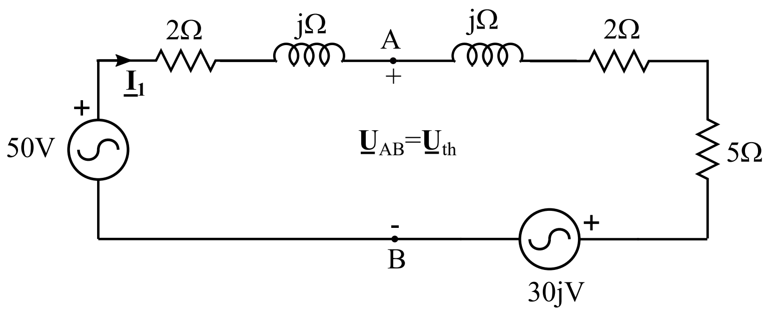


Solution

As we want to find the current i(t), we should calculate the Thevenin equivalent of the circuit in the frequency domain excluding the central branch of the circuit:



1. First we calculate the phasor Thevenin voltage, which is the phasor voltage between terminals A and B:



$$\underline{\mathbf{I}}_1 = \frac{50 - 30j}{2 + j + j + 2 + 5} = 4.59 - 4.35jA$$

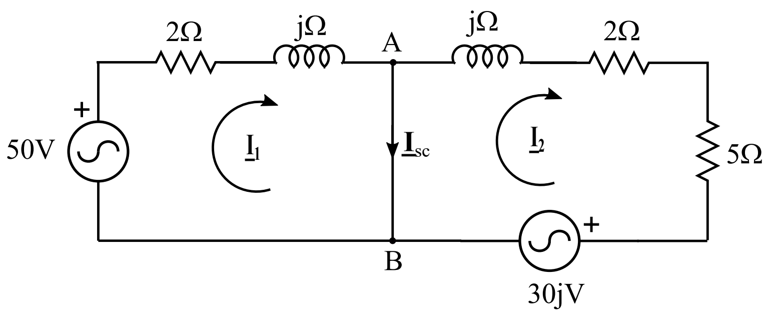
$$\underline{\mathbf{U}}_{th} = \underline{\mathbf{U}}_{AB} = (7 + j) \cdot \underline{\mathbf{I}}_1 + 30j = 36.47 + 4.12jV$$

2. Calculation of Thevenin impedance

   To calculate Z_{th} we have two alternative methods: The calculation of the short-circuit current and the calculation of the equivalent resistance of the passive circuit between terminals A and B.

   (a) Method 1: Calculation of $\underline{\mathbf{I}}_{sc}$

   We place a short-circuit between A and B and calculate the phasor current ($\underline{\mathbf{I}}_{sc}$) flowing from A to B.



   The mesh equations of the circuit can be written in matrix form:

$$\begin{pmatrix} 2+j & 0 \\ 0 & 7+j \end{pmatrix} \cdot \begin{pmatrix} \underline{\mathbf{I}}_1 \\ \underline{\mathbf{I}}_2 \end{pmatrix} = \begin{pmatrix} 50 \\ -30j \end{pmatrix}$$

$$\underline{\mathbf{I}}_1 = 20 - 10jA$$

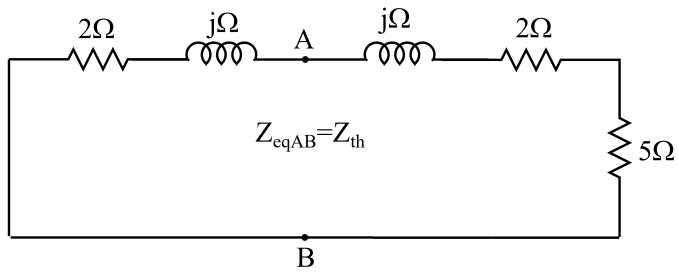
$$\underline{\mathbf{I}}_2 = -0.6 - 4.2jA$$

$$\underline{\mathbf{I}}_{sc} = \underline{\mathbf{I}}_1 - \underline{\mathbf{I}}_2 = 20.6 - 5.8jA$$

$$Z_{th} = \frac{\underline{\mathbf{U}}_{th}}{\underline{\mathbf{I}}_{sc}} = 1.59 + 0.65j\Omega$$

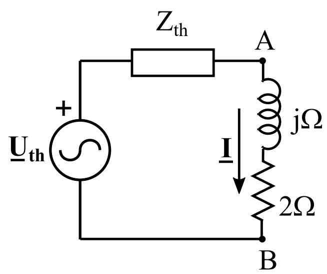
(b) Method 2: Calculation of $Z_{eq_{AB}}$

We passivize the system turning off the sources and we obtain the following net of impedances:



$$Z_{th} = Z_{eq_{AB}} = (7+j)||(2+j) = 1.59 + 0.65j\Omega$$

Finally, we connect the central branch that we want to study between terminals AB of the equivalent and calculate the phasor current ($\underline{\mathbf{I}}$)



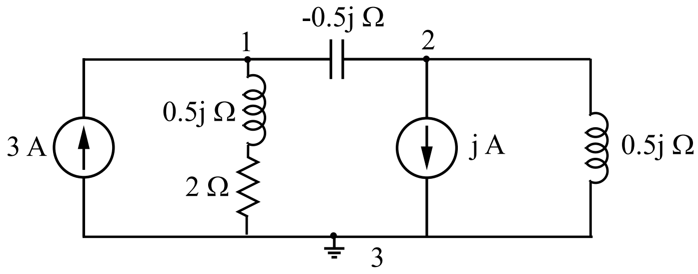
$$\underline{\mathbf{I}} = \frac{\underline{\mathbf{U}}_{th}}{Z_{th} + Z_{central}} = 8.83 - 2.91jA = 9.3\angle{-18.4^o}A$$

$$i(t) = \sqrt{2} \cdot 9.3 \cdot \cos(1000 \cdot t - 18.4^o)A$$

Which is the same result that we obtained when we solved the circuit with mesh analysis.

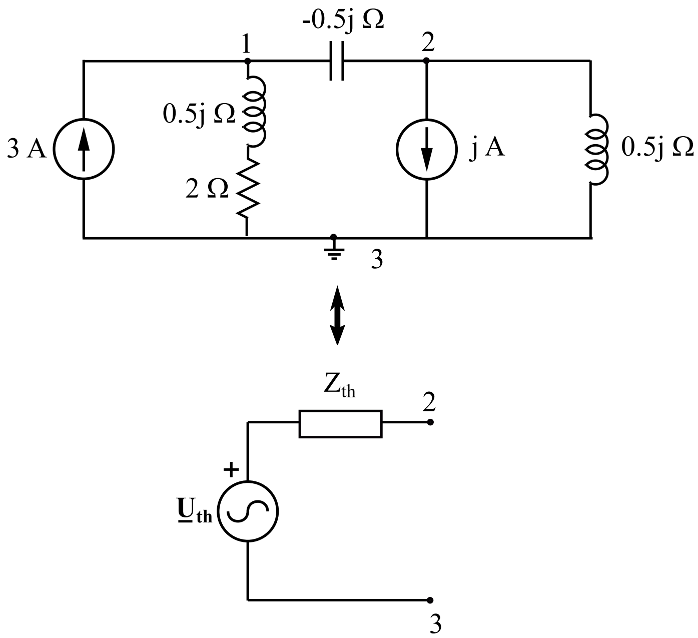
7.4.2 Example 2

Find the Thevenin's equivalent of the circuit of the example in section 7.3.2 between terminals 2 and 3.

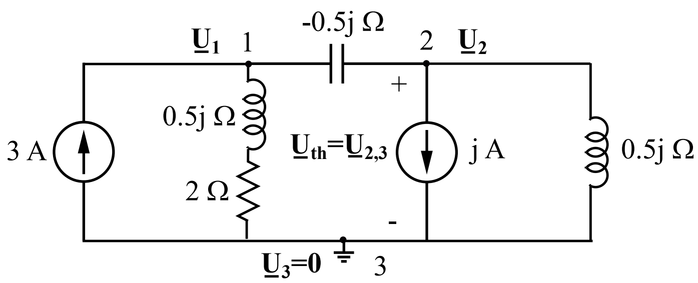


Solution

We want to include all the elements of the circuit in the equivalent.



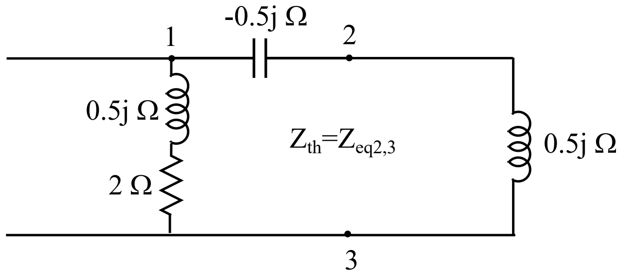
1. Calculation of Thevenin's voltage



$$\underline{\mathbf{U}}_{th} = \underline{\mathbf{U}}_{2,3} = \underline{\mathbf{U}}_2 - \underline{\mathbf{U}}_3 = 0,47 + 1,38j = 1,46\angle 71,2^o V$$

2. Calculation of Zth

   We passivize the circuit and calculate the equivalent impedance between terminals 2-3



$$Z_{th} = (2 + 0.5j - 0.5j)||0.5j = \frac{2 \cdot 0.5j}{2 + 0.5j} = 0.12 + 0.47j = 0.48\angle 75.96^o\Omega$$

   We would obtain the same impedance applying the short-circuit current method.

7.5 Analysis of circuits with sources of several frequencies

7.5.1 Superposition principle

To analyse a circuit in with sources of different frequencies act simultaneously, the transformation of the circuit to the frequency domain can not be applied because: which frequency is considered for the calculation of the impedances?

To solve this case we can apply the **superposition principle**, obtain the response of the circuit to the effect of each source separately and sum up the individual responses to find the total response of the circuit.

Remember that the superposition principle states that "The response of a linear circuit subjected to several excitation sources acting simultaneously equals the sum of the responses of the circuits when the sources act separately".

Then if we want to solve a circuit that incorporates sources of two different frequencies, ω_1 and ω_2, we will proceed as follows:

1. Turn off all the sources of frequency ω_2 replacing the voltage sources by short circuits and the current sources by open circuits.

2. Transform the circuit into the frequency domain considering the frequency ω_1 to calculate the impedances of the passive elements.

3. Calculate the response of the circuit in the frequency domain (i.e. the phasors current and voltage at the different parts of the circuit) and then find the instantaneous currents and voltages in the time domain which constitute the response of the circuit to the sources of frequency ω_1. Note that those instantaneous currents and voltages are sinusoidal functions of frequency ω_1.

4. Turn off all the sources of frequency ω_1 replacing the voltage sources by short circuits and the current sources by open circuits.

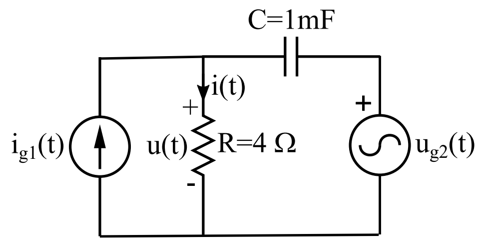
5. Recalculate the impedances of the passive elements for the frequency ω_2 and obtain the currents and voltages of the elements in the frequency domain and in the time

domain. These currents and voltages are the response of the circuit to the sources of frequency ω_2 and are sinusoidal functions of frequency ω_2.

6. Obtain the response of the system when all the sources act simultaneously as the sum of the separate responses in the time domain. The currents and voltages will be a sum of two sinusoidal functions of frequencies ω_1 and ω_2.

7.5.2 Example

Calculate the current i(t) and the voltage u(t) in the following circuit:

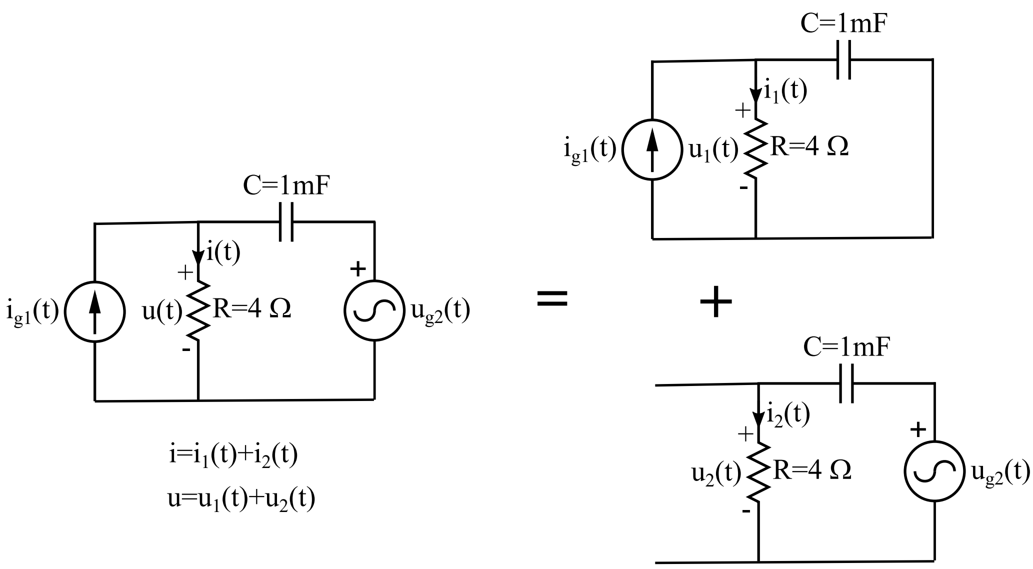


$$i_{g1}(t) = \sqrt{2} \cdot 10 \cdot \cos(100 \cdot t + 90)V$$
$$u_{g2}(t) = \sqrt{2} \cdot 50 \cdot \cos(200 \cdot t - 90)V$$

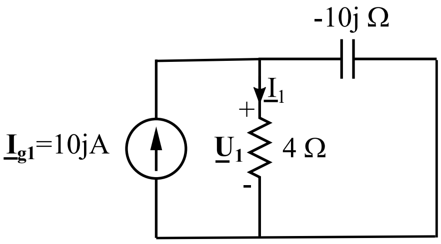
Solution

As the two sources have different frequency, we need to apply superposition principle to solve the circuit.

First we will cancel the voltage source $u_{g2}(t)$ and analyse the response of the circuit when only the current source $i_{g1}(t)$ is applied. Then we will cancel the current source and obtain the response of the circuit to the voltage source. The total response is calculated as the sum of the individual responses.



1. Response to $i_{g1}(t)$.



The frequency of $i_{g1}(t)$ is 100 rad/s, then the impedance of the capacitor is $-10j\Omega$. We apply the current divider equation to calculate $\underline{\mathbf{I}}_1$

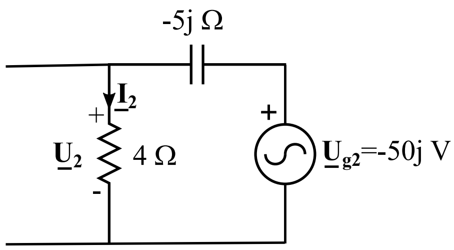
$$\underline{\mathbf{I}}_1 = \underline{\mathbf{I}}_{g1} \cdot \frac{Z_C}{Z_R + Z_C} = 10j \cdot \frac{-10j}{4-10j} = 3.45 + 8.62jA = 9.28\angle 68.20^o A$$

$$\underline{\mathbf{U}}_1 = \underline{\mathbf{I}}_1 \cdot Z_R = 13.79 + 34.48jV = 37.14\angle 68.20^o V$$

That in the time-domain are:

$$i_1(t) = \sqrt{2} \cdot 9.28 \cdot \cos(100 \cdot t + 68.20^o)A$$
$$u_1(t) = \sqrt{2} \cdot 37.14 \cdot \cos(200 \cdot t + 68.20^o)V$$

2. Response to $u_{g2}(t)$



As the frequency of $u_{g2}(t)$ is 200 rad/s, the impedance of the capacitor is $-5j\Omega$. We apply Ohm's law in the frequency domain to calculate $\underline{\mathbf{I}}_2$:

$$\underline{\mathbf{I}}_2 = \frac{\underline{\mathbf{U}}_{g2}}{Z_R + Z_C} = \frac{-50j}{4-10j} = 6.10 - 4.88jA = 7.81\angle -38.66A$$

$$\underline{\mathbf{U}}_2 = \underline{\mathbf{I}}_2 \cdot Z_R = 24.39 - 19.51jV = 31.23\angle -38.66^o V$$

That in the time-domain are:

$$i_2(t) = \sqrt{2} \cdot 7.81 \cdot \cos(100 \cdot t - 38.66^o)A$$
$$u_2(t) = \sqrt{2} \cdot 31.23 \cdot \cos(200 \cdot t - 38.66^o)V$$

3. We calculate the response to both fonts acting simultaneously as the sum of individual responses in the time domain:

$$i(t) = i_1(t) + i_2(t) = \sqrt{2} \cdot 9.28 \cdot \cos(100 \cdot t + 68.20^o) + \sqrt{2} \cdot 7.81 \cdot \cos(100 \cdot t - 38.66^o) A$$

$$u(t) = u_1(t) + u_2(t) = \sqrt{2} \cdot 37.14 \cdot \cos(200 \cdot t + 68.20^o) + \sqrt{2} \cdot 31.23 \cdot \cos(200 \cdot t - 38.66^o) V$$

7.6 Analysis of circuits with coupled inductors

7.6.1 Impedance of coupled inductors

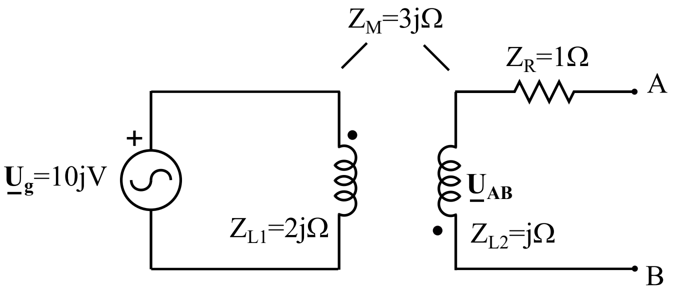
Circuits with coupled inductors can also be analysed in the frequency domain applying the methods studied in previous sections. The impedance for the mutual inductance coefficient is calculated as:

$$Z_M = j\omega M \tag{102}$$

To calculate the voltage drop across the inductors we need to consider the effect of the self inductance and the effect of the mutual inductance. The polarity of the voltages is derived from the analysis of the dotted terminals.

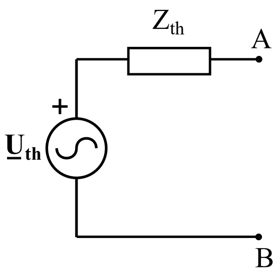
7.6.2 Example

Given the following circuit in the frequency domain, obtain the Thevenin's equivalent between terminals AB:

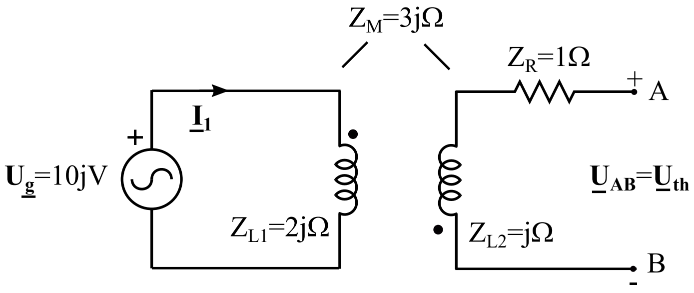


Solution

We want to find the parameters $\underline{\mathbf{U}}_{th}$ and Z_{th} that makes the following circuit equivalent to the initial circuit between terminals A B:



Firstly we calculate Thevenin's voltage, which is the voltage drop between A and B in open circuit:



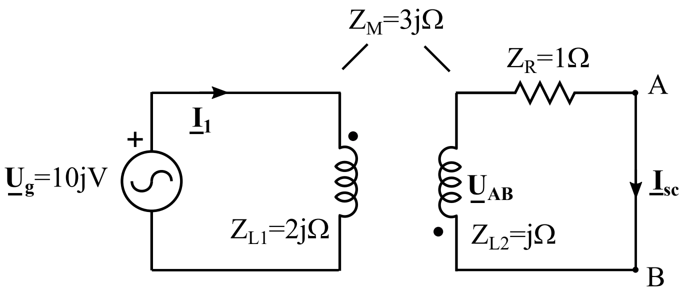
The mesh equation for the circuit at the left is:

$$-\underline{\mathbf{U}}_g + \underline{\mathbf{I}}_1 \cdot Z_{L1} = 0$$

$$\underline{\mathbf{I}}_1 = \frac{\underline{\mathbf{U}}_g}{Z_{L1}} = \frac{10j}{2j} = 5A$$

$$\underline{\mathbf{U}}_{th} = \underline{\mathbf{U}}_{AB} = -Z_M \cdot \underline{\mathbf{I}}_1 = -5 \cdot 3j = -15jV$$

Now we calculate Thevenin's impedance. To this end we place a short circuit between A and B and calculate the current flowing from A to B:



Applying 2KL to the circuit at the left and the circuit at the right:

$$-\underline{\mathbf{U}}_g + \underline{\mathbf{I}}_1 \cdot Z_{L1} + Z_M \cdot \underline{\mathbf{I}}_{sc} = 0$$

$$Z_{L2} \cdot \underline{\mathbf{I}}_{sc} + Z_R \cdot \underline{\mathbf{I}}_{sc} + Z_M \cdot \underline{\mathbf{I}}_1 = 0$$

Substituting numerical values:

$$-10j + \underline{\mathbf{I}}_1 \cdot 2j + 3j \cdot \underline{\mathbf{I}}_{sc} = 0$$

$$(j+1) \cdot \underline{\mathbf{I}}_{sc} + 3j \cdot \underline{\mathbf{I}}_1 = 0$$

Solving the system:

$$\underline{\mathbf{I}}_1 = -0.94 + 1.70jA \qquad \underline{\mathbf{I}}_{sc} = 3.96 - 1.13jA$$

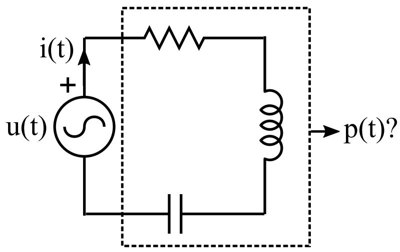
Then Thevenin's impedance is:

$$Z_{th} = \frac{\underline{\mathbf{U}}_{th}}{\underline{\mathbf{I}}_{sc}} = \frac{-15j}{3.96 - 1.13j} = 1 - 3.5j\Omega$$

8 Power in AC circuits

8.1 Instantaneous power in AC

Imagine that we want to calculate the instantaneous power absorbed by the RLC net in the following circuit:



The voltage and the current are sinusoidal of the same frequency:

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \tag{103}$$

$$i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i) \tag{104}$$

being:

$$\varphi = \varphi_u - \varphi_i \tag{105}$$

To simplify the analysis of the power, it is more convenient to take the current as phase origin, then the expressions for the current and voltage would be:

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi) \tag{106}$$

$$i(t) = \sqrt{2} \cdot I \cdot \cos \omega t \tag{107}$$

The instantaneous power is the product of the voltage and the current:

$$p(t) = u(t) \cdot i(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi) \cdot \sqrt{2} \cdot I \cdot \cos \omega t \tag{108}$$

using the trigonometric relation:

$$\cos \alpha \cdot \cos \beta = \frac{1}{2} \cdot (\cos(\alpha + \beta) + \cos(\alpha - \beta)) \tag{109}$$

$$p(t) = U \cdot I \cdot cos\varphi + U \cdot I \cdot \cos(2\omega t + \varphi) \tag{110}$$

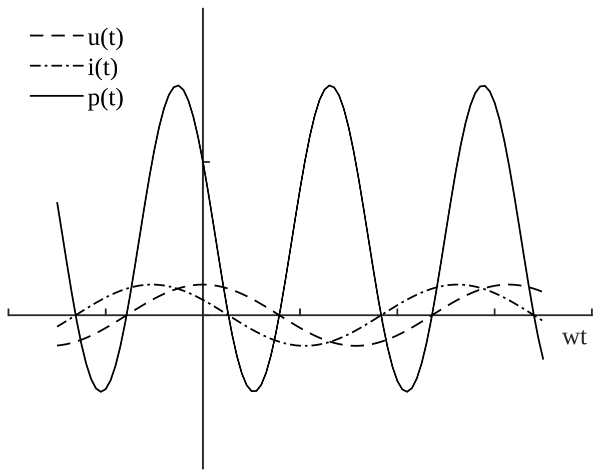
and now applying:

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \tag{111}$$

the power results:

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t \tag{112}$$

If the instantaneous power, the voltage and the current are plotted vs. time, we see that power is a sinusoidal function whose frequency doubles that of voltage and current. As can be seen, in some parts of the cycle the power becomes negative. This means that the RLC net sometimes absorbs power (when the power is positive) but at other instants delivers power (when the power is negative).

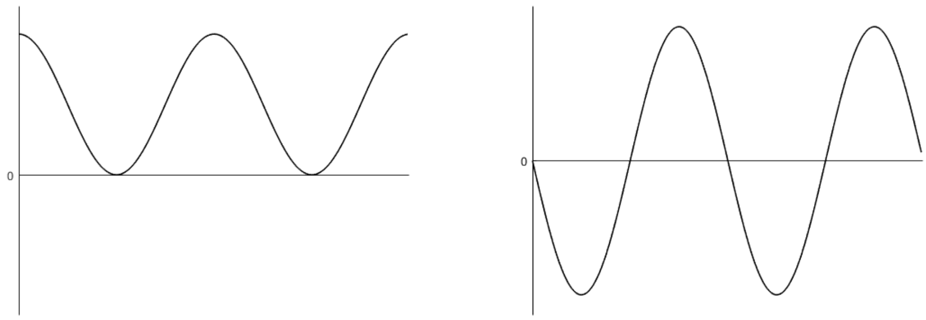


It is interesting to analyse the two terms of the power separately to understand what is the meaning of equation (112):

$$p(t) = \underbrace{U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t)}_{1} - \underbrace{U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t}_{2}$$

If terms 1 and 2 are plotted vs. time we see that the first part of the power correspond to a power that is always positive (absorbed).

$U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t)$ $\qquad\qquad$ $-\, U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$



As will be justified next, the "positive power" corresponds to the power dissipated in the resistor. The average value of this power is $U \cdot I \cdot \cos\varphi$. This expression is also the average value of the instantaneous power. On the other hand, the "fluctuating power" corresponds to the exchange of power that takes place between the source and the capacitor and inductor. The average value of the fluctuating power is zero and its amplitude $U \cdot I \cdot \sin\varphi$.

We define two new terms in relation to power in AC circuits:

- **Active power**: Is the average value of the instantaneous power

$$P = U \cdot I \cdot \cos\varphi \tag{113}$$

  Later it will be justified that the active power represents the power absorbed in the resistances.

- **Reactive power**: Is the amplitude of the fluctuating power:

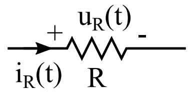
$$Q = U \cdot I \cdot \sin\varphi \tag{114}$$

  As we will see in the next sections, the reactive power represents the power exchanged between the inductors and capacitors and the sources.

To deepen into the meaning of the instantaneous power variation and the active and reactive power we will analyse the instantaneous power of resistors, inductors and capacitors separately.

8.2 Power of a resistor

We want to calculate the value of the power absorbed by a resistor of value R when sinusoidal current i_R flows through it and its voltage drop is u_R.



As was studied before, the phase shift between the voltage and the current across a resistor is zero:

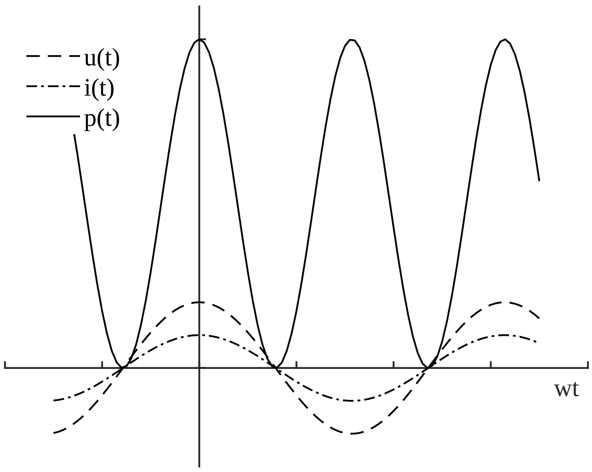
$$\varphi_R = \varphi_u - \varphi_i = 0 \tag{115}$$

Then, the instantaneous power is:

$$p_R(t) = U \cdot I \cdot (1 + \cos(2\omega t)) \tag{116}$$

If we represent the instantaneous power vs. time we see that:

- The instantaneous power of a resistor is always positive. This is consistent with what we studied before, because resistors **always** dissipate energy.
- The power fluctuates with frequency 2ω
- The average value of the power is $U \cdot I$



The active and reactive power of a resistor are:

$$P_R = U \cdot I \cdot \cos\varphi_R = U \cdot I \tag{117}$$

$$Q_R = U \cdot I \cdot \sin\varphi_R = 0 \tag{118}$$

As can be seen, the reactive power in the resistor is zero, since in this element there is no fluctuation of energy and all the power absorbed from the source is dissipated.

Additionally, considering the relation between the phasor voltage and the phasor current across a resistor and the relation verified by the modulus of the two phasors:

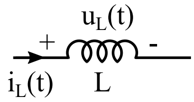
$$\underline{\mathbf{U}} = R \cdot \underline{\mathbf{I}} \qquad\qquad U = R \cdot I$$

We may find two additional expressions that can be used to calculate the active power of a resistor as a function of the voltage or the current and the resistance:

$$P_R = R \cdot I^2 = \frac{U^2}{R} \tag{119}$$

8.3 Power of an inductor

As with the resistance, we are going to calculate the power of an inductor with current flow i_L and voltage drop u_L:



As was studied before, the phase shift between the voltage and the current across an inductor is 90^o:

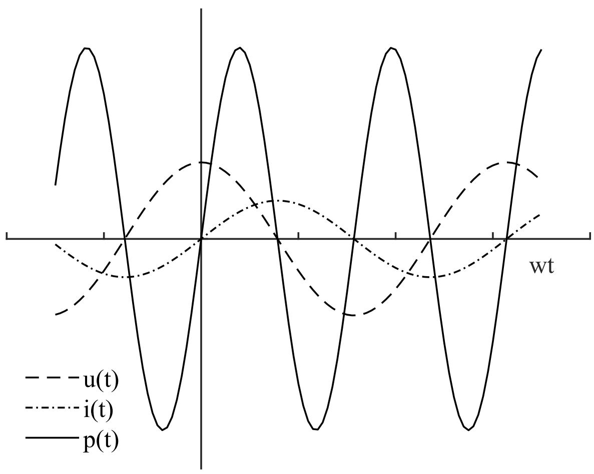
$$\varphi_L = \varphi_u - \varphi_i = 90^o$$

Then, the instantaneous power is:

$$p_L(t) = -U \cdot I \cdot \sin(2\omega t)$$

If we represent the power we see that:

- The power fluctuates with frequency 2ω being sometimes positive and sometimes negative. This means that the element at some times absorbs power and at other times delivers power.
- The average value of the instantaneous power is zero.



These observations are consistent with what studied about power of inductors: Inductors do not dissipate power, but they store energy in a magnetic field. As can be seen in the graph in one half of the cycle the energy is stored, and at the next part of the cycle the energy is released and returned to the source.

The active and reactive power of an inductor are:

$$P_L = U \cdot I \cdot \cos\varphi_L = 0 \tag{120}$$

$$Q_L = U \cdot I \cdot \sin\varphi_L = U \cdot I \tag{121}$$

The active power of an inductor is 0, since P represents an energy dissipation which does not occur in ideal inductors.

Additionally, considering the relation between the phasor voltage and the phasor current across an inductor, and the relation verified by the modulus of the two phasors:

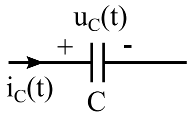
$$\underline{\mathbf{U}} = j\omega L \cdot \underline{\mathbf{I}} \qquad\qquad U = \omega \cdot L \cdot I = X_L \cdot I$$

We may find two additional expressions that can be used to calculate the reactive power of an inductor as a function of the voltage or the current and the reactance of the inductor:

$$Q_L = X_L \cdot I^2 = \frac{U^2}{X_L} \tag{122}$$

The reactive power of an inductor is always positive; we say that **inductors always absorb reactive power**.

8.4 Power of a capacitor



As was studied before, the phase shift between the voltage and the current across a capacitor is -90^o:

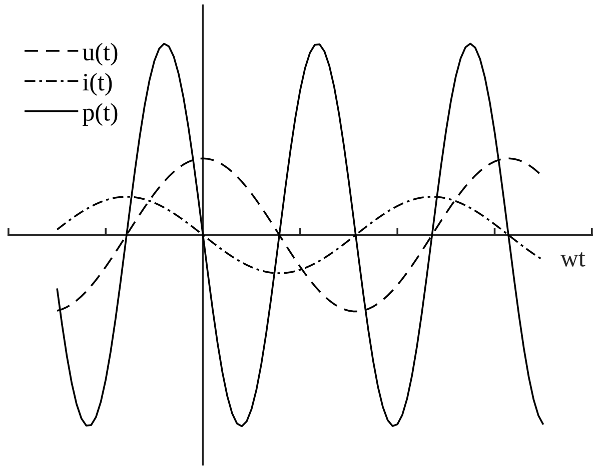
$$\varphi = \varphi_u - \varphi_i = -90^o$$

Then, the instantaneous power is:

$$p_C(t) = U \cdot I \cdot \sin(2\omega t)$$

If we represent the power we see a similar behaviour to the one of the inductor:

- The power fluctuates with frequency 2ω being sometimes positive and sometimes negative, what means that the capacitor absorbs and delivers power alternatively.
- The average value of the instantaneous power is zero.



The observations are consistent with the fact that capacitors do not dissipate power, but store energy in an electric field instead. What we see in the graph is that in the first part of the cycle the energy is stored and then the energy is released and returned to the source.

The active and reactive power of a capacitor are:

$$P_C = U \cdot I \cdot \cos\varphi_C = 0 \tag{123}$$

$$Q_C = U \cdot I \cdot \sin\varphi_C = -U \cdot I \tag{124}$$

Additionally, considering the relation between the phasor voltage and the phasor current across a capacitor, and the relation verified by the modulus of the two phasors:

$$\underline{\mathbf{U}} = \frac{-j}{\omega \cdot C} \cdot \underline{\mathbf{I}} \qquad\qquad U = \frac{1}{\omega \cdot C} \cdot I = -X_C \cdot I$$

We may find two additional expressions that can be used to calculate the reactive power of a capacitor as a function of the voltage or the current and the reactance ($X_C = -\frac{1}{\omega \cdot C}$):

$$Q_C = X_C \cdot I^2 = \frac{U^2}{X_C} \tag{125}$$

The reactive power of a capacitor is always negative; we say that **capacitors always absorb reactive power**.

8.5 Definitions

We can distinguish two different behaviours in relation to energy in AC circuits:

1. Resistors absorb power from the sources and transform it into heat.
2. Inductors and capacitors absorb power from the sources, store it and return it later.

The absorbed power and the fluctuating power are both important and must be taken into account in the analysis of AC circuits. Nevertheless they have a different impact on circuits: as the first supposes a real consumption of energy the second is a continuous flow of energy between the source and the inductive and capacitive loads.

We are have defined two types of power in AC systems:

- **Active power (P)**: is power absorbed in resistors. From the mathematical point of view, the active power of a system can be calculated as:

$$P = U \cdot I \cdot \cos\varphi \tag{126}$$

  Active power is measured in Watts [W]

- **Reactive power (Q)**: is the power that fluctuates between the inductors and capacitors and the sources.

$$Q = U \cdot I \cdot \sin\varphi \tag{127}$$

  Reactive power is measured in Volt-Ampere reactive [var]

The instantaneous power can be also expressed in terms of the active and reactive power as:

$$p(t) = P \cdot (1 + cos 2\omega t) - Q \cdot \sin 2\omega t \tag{128}$$

Additionally other concepts are important for the analysis of the power absorbed and delivered in AC circuits:

- **Power factor**: is the cosine of the phase shift between voltage and current.

$$p.f. = \cos\varphi = \cos(\varphi_u - \varphi_i) \tag{129}$$

- **Complex power ($\boldsymbol{S}$)** Is the complex sum of the active and reactive power.

$$\mathcal{S} = P + Q = U \cdot I \cdot \cos\varphi + j \cdot U \cdot I \cdot \sin\varphi = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* \tag{130}$$

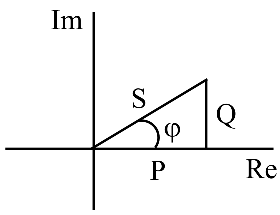
  Complex power is measured in Volt-Ampere [VA]

- **Apparent power (S)**: Is the modulus of the complex power:

$$S = \sqrt{P^2 + Q^2} = U \cdot I \tag{131}$$

  Apparent power is measured in Volt-Ampere [VA].

- **Power triangle**: We can plot the active and reactive power in the so-called power triangle, which is a representation of the complex power in the complex plane.

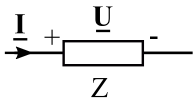


As it can be seen, the angle φ of the power triangle is the same as the phase shift between voltage and current.

8.6 Power of a complex impedance

If we consider a complex impedance:

$$Z = R + j \cdot X \tag{132}$$



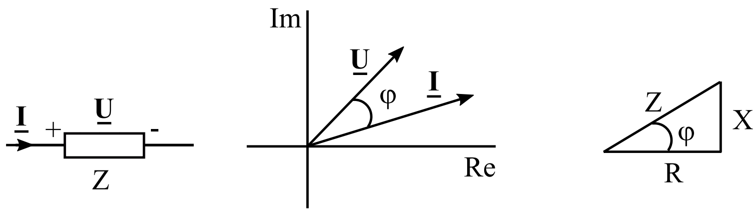
The relation between the phasor voltage and current is given by Ohm's law:

$$\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}} \tag{133}$$

Note that the angle φ is the phase difference between the phasor voltage and the phasor current, but it is also the angle of the complex impedance.

$$Z = \frac{\underline{\mathbf{U}}}{\underline{\mathbf{I}}} = \frac{U\angle\varphi_u}{I\angle\varphi_i} = \frac{U}{I}\angle\varphi_u - \varphi_i = \frac{U}{I}\angle\varphi \tag{134}$$

If the impedance is represented as an impedance triangle we see that the same angle φ appears between the voltage and current and between the components of the complex impedance:



The active and reactive and complex power absorbed by the impedance are:

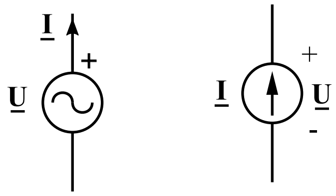
$$P = U \cdot I \cdot \cos\varphi = R \cdot I^2 \tag{135}$$

$$Q = U \cdot I \cdot \sin\varphi = X \cdot I^2 \tag{136}$$

$$\mathcal{S} = P + Q = Z \cdot I^2 \tag{137}$$

8.7 Power of an AC source

To calculate the power delivered by an AC source we adopt the sign criteria defined for DC circuits: A source delivers power when current flows from the lower voltage terminal towards the higher voltage terminal. The power delivered by a source is taken as positive.



The complex power generated by an AC source is calculated as:

$$\mathcal{S}_g = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* = P_g + Q_g j \tag{138}$$

The real part of the complex power is the active power delivered by the source and the imaginary part is the reactive power delivered by the source. In some cases P_g or Q_g might be negative, what means that the source absorbs active or reactive power.

*** Proof for equation (138)**:

$$\underline{\mathbf{U}} = U\angle\varphi_u \qquad \underline{\mathbf{I}} = I\angle\varphi_i \qquad \varphi = \varphi_u - \varphi_i$$

$$\mathcal{S}_g = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* = U\angle\varphi_u \cdot I\angle{-\varphi_i} = U \cdot I\angle\varphi = U \cdot I\cos\varphi + j \cdot U \cdot I\sin\varphi = P_g + j \cdot Q_g$$

8.8 Boucherot's Theorem

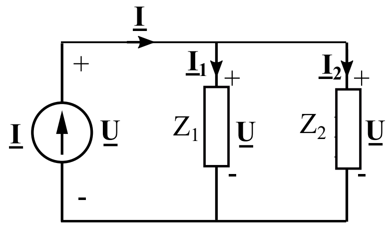
Boucherot's Theorem establishes that the total amount of active and reactive power absorbed in an electric circuit equals the sum of the active and reactive power absorbed by its passive elements.

$$P_T = \sum_k P_k \tag{139}$$

$$Q_T = \sum_k Q_k \tag{140}$$

This principle allows us to derive that in AC circuits there is a power balance in which the complex power supplied by the sources equals the complex power absorbed by the passive elements.

For example in the circuit of the figure:

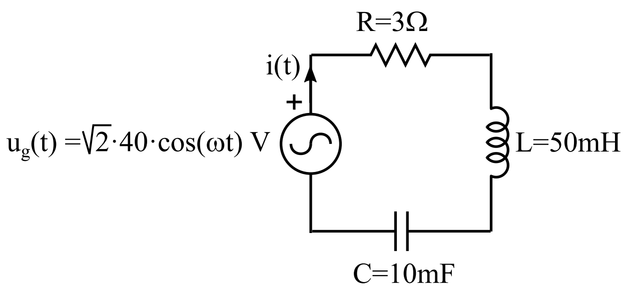


The power supplied by the source equals the sum of the complex power absorbed by the impedances Z_1 and Z_2.

$$\mathcal{S}_{\}} = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* = \underline{\mathbf{U}} \cdot (\underline{\mathbf{I}}_1 + \underline{\mathbf{I}}_2)^* = S_{Z_1} + S_{Z_2} = (P_1 + P_2) + j(Q_1 + Q_2)$$

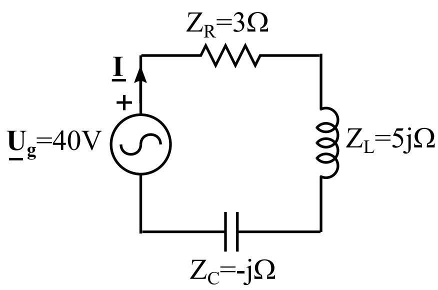
8.9 Example

Do a power balance of the following circuit:



Solution

The analysis of the circuit was carried out in detail in Section 6.8, where the circuit was solved in the frequency domain:



The phasor current $\underline{\mathbf{I}}$ is:

$$\underline{\mathbf{I}} = \frac{\underline{\mathbf{U}}_g}{Z_R + Z_L + Z_C} = \frac{40}{3 + 5j - j} = 8\angle - 53.13^o A$$

1. Power absorbed by the passive elements:

- Resistor:

  The resistor only absorbs active power

$$P_R = R \cdot I^2 = 3 \cdot 8^2 = 192W$$

$$Q_R = 0var$$

- Inductor:

  The inductor only absorbs reactive power

$$P_L = 0W$$

$$Q_L = X_L \cdot I^2 = 5 \cdot 8^2 = 320var$$

- Capacitor:

  The capacitor delivers reactive power

$$P_C = 0W$$

$$Q_C = X_C \cdot I^2 = -1 \cdot 8^2 = -64var$$

Then, the complex power absorbed by the loads is

$$\mathcal{S}_{loads} = P_R + j \cdot (Q_L + Q_C) = 192 + j(320 - 64) = 192 + 256jVA$$

2. Power delivered by the source:

$$\mathcal{S}_g = \underline{\mathbf{U}}_g \cdot \underline{\mathbf{I}}^* = 40 \cdot 8\angle 53.13^o = 320\angle 53.13^o = 192 + 256jVA$$

As can be seen the power absorbed by the loads equals the power delivered by the source: $\mathcal{S}_g = \mathcal{S}_{loads}$