Module 3: Analysis of AC circuits

Electrical power engineering fundamentals

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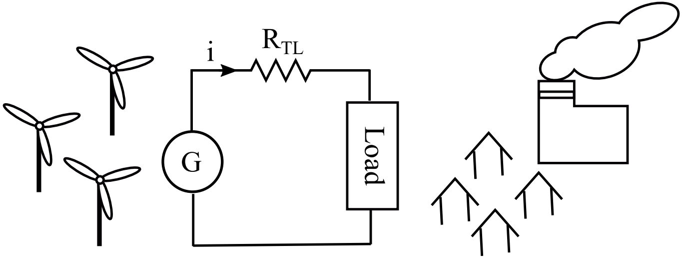
Why AC systems



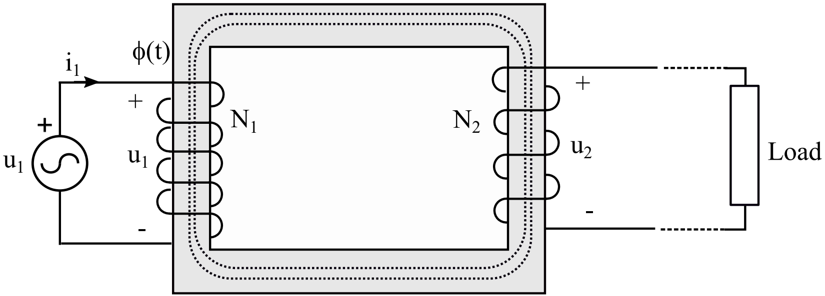
The line losses depend on the current that flows through the line.

$$p_{load} = u \cdot i \qquad p_{loss} = R_{TL} \cdot i^2$$

If the voltage of the system is raised up the line losses cuts

Transformers

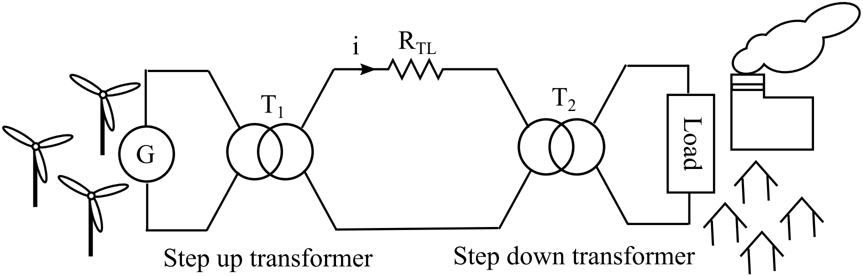
Transformers are used to change the voltage level of electric energy.



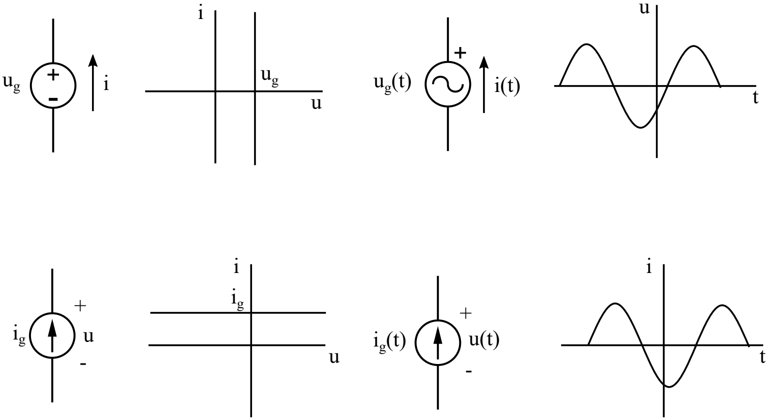
Transformers are based in two coupled inductors with different number of turns. The **transformation relation** is:

$$r_t = \frac{N_1}{N_2} = \frac{u_1}{u_2}$$

Typical configuration of a power system

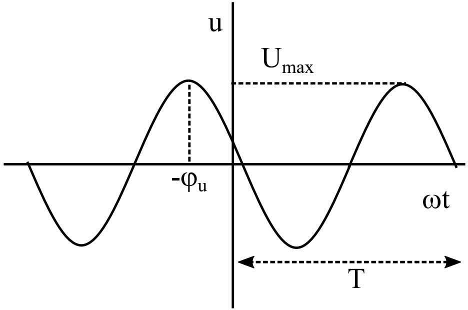


DC vs. AC sources



$$u(t) = U_{max} \cdot \cos(\omega t + \varphi_u) \qquad i(t) = I_{max} \cdot \cos(\omega t + \varphi_i)$$

Main parameters of a sinusoidal signal



$$u(t) = U_{max} \cdot \cos(\omega t + \varphi_u)$$

Sinusoidal functions might also be defined in terms of a function sine ($sin\alpha = \cos(\alpha - \pi/2)$) but cosine is used in this course

Main parameters of a sinusoidal signal

- **Amplitude** (Umax): maximum value reached by the voltage in the whole function range
- **Period** (T): time needed to complete a cycle (expressed in [s])
- **Frequency**: number of cycles described in one second.

$$f = \frac{1}{T} \quad [Hz]$$

- **Angular frequency**: Frequency of the function in radians per second.

$$\omega = 2 \cdot \pi \cdot f \quad [rad] \cdot [s]^{-1}$$

- **Phase angle** (φ): phase difference between the maximum of the function and the origin (expressed in [rad] and sometimes in degrees).

Main parameters of a sinusoidal signal

- **Mean value**: The mean value of a sinusoidal function equals zero

$$U_{mean} = \frac{1}{T}\int_{t_0}^{t_0+T} u(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} U_{max}\cdot\cos(\omega t+\varphi_u)dt = 0$$

- **Root mean square value** (rms) or effective value:

$$U_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} u^2(t)dt}$$

RMS value of a sinusoidal signal

The RMS value of a sinusoidal signal is:

$$U_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} U_{max}^2 \cdot \cos^2(\omega t + \varphi_u) dt} = \frac{U_{max}}{\sqrt{2}}$$

The rms value of the sinusoidal signals is a key parameter and it is functions in terms of rms instead of peak values:

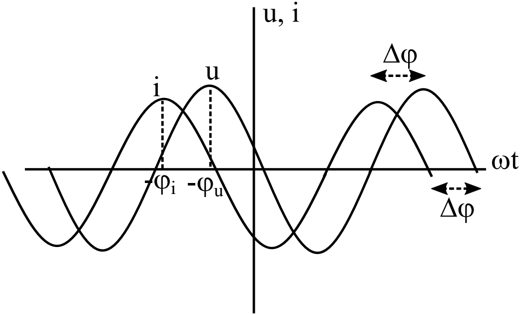
$$U = U_{rms} = \frac{U_{max}}{\sqrt{2}} \qquad I = I_{rms} = \frac{I_{max}}{\sqrt{2}}$$

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i)$$

Relative phase shift between two signals

Distance between the zero crossings or the peaks of the signals

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i)$$

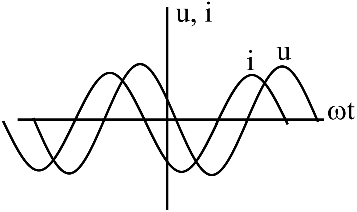


$$\Delta\varphi_{u,i} = \varphi_u - \varphi_i$$

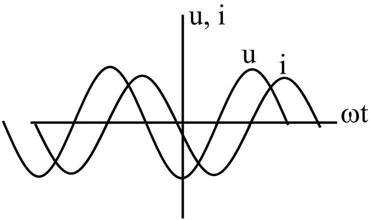
$\Delta\varphi_{u,i} < 0$: voltage lags current and current leads voltage.

$\Delta\varphi_{u,i} > 0$: current lags voltage and voltage leads current.

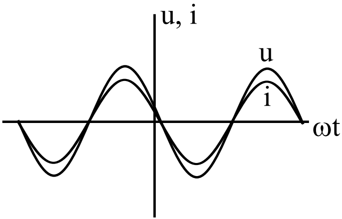
Relative phase shift between two signals



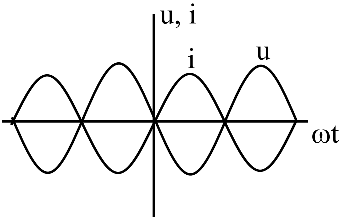
Current leads voltage
Voltage lags current



Voltage leads current
Current lags voltage



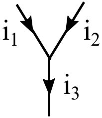
Current and voltage are in phase



Current and voltage are in antiphase

Challenges of the analysis of AC circuits in the time domain

1. Operating with sinusoidal functions is not easy.



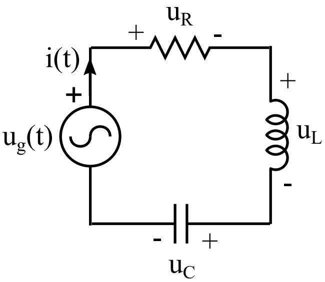
$i_1 + i_2 = i_3$

$$i_3 = \sqrt{2} \cdot I_1 \cdot \cos(\omega t + \varphi_{i_1}) + \sqrt{2} \cdot I_2 \cdot \cos(\omega t + \varphi_{i_2}) = \sqrt{2} \cdot I_3 \cdot cos(\omega t + \varphi_{i_3})$$

   Finding I_3 and φ_{i_3} requires complicated math analysis

2. The analysis of AC circuits involves the solution of differential equations or differential systems of equations.

Analysis of a RLC circuit in the time domain

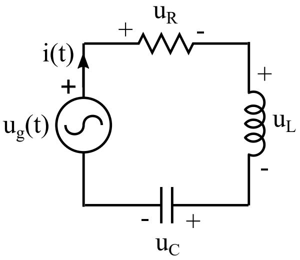


$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u)$$

According to KVL: $u_g(t) = u_R(t) + u_L(t) + u_C(t)$

$$u_R(t) = R \cdot i(t) \qquad u_L(t) = L \cdot \frac{di(t)}{dt} \qquad u_C(t) = \frac{1}{C} \cdot \int i(t)dt$$

Analysis of a RLC circuit in the time domain



$$u_g(t) = R \cdot i(t) + L \cdot \frac{di(t)}{dt} + \frac{1}{C} \cdot \int i(t)dt$$

$$\frac{du_g(t)}{dt} = R \cdot \frac{di(t)}{dt} + L \cdot \frac{d^2 i(t)}{dt^2} + \frac{1}{C} \cdot i(t)$$

$$i(t) = i_{transient} + i_{steady-state}$$

$$u_g(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \quad => \quad i_{ss}(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i)$$

Some conclusions about the analisis of AC circuits

- The analysis of AC circuits involves the resolution of differential equations or systems of equations. For complex circuits finding and a solution is challenging.
- If the excitation of a circuit is a sinusoidal voltage or current of frequency ω, all the resulting currents and voltages are also sinusoidal functions of the same frequency.
- Our goal is to find the amplitudes and phase shifts of the responses.
- Below we introduce the analysis of AC circuits in the frequency domain which is based in the representation of sinusoidal functions by means of complex numbers.

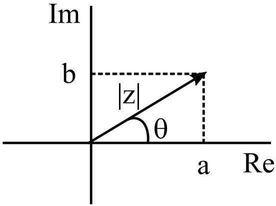
Complex numbers: a short review

$$j = \sqrt{-1}$$

Complex number:

$$z = a + bj$$

$$z = |z|\angle\theta \qquad z = |z| \cdot e^{j\theta}$$



Conversion between rectangular and polar form:

$$|z| = \sqrt{a^2 + b^2} \quad \theta = \arctan\frac{b}{a} \qquad a = |z| \cdot \cos\theta \quad b = |z| \cdot \sin\theta$$

Euler's equation:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Phasor representation of a sinusoidal function

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u)$$

Applying Euler's equation ($e^{j\theta} = \cos\theta + j\sin\theta$), u(t) can be expressed as:

$$u(t) = \sqrt{2} \cdot U \cdot Re(e^{j(\omega t + \varphi_u)}) = \sqrt{2} \cdot Re(U \cdot e^{j\varphi_u} \cdot e^{j\omega t})$$

The important information of u(t) is U and φ_u, since ω is the same for all the currents and voltages of the circuit.

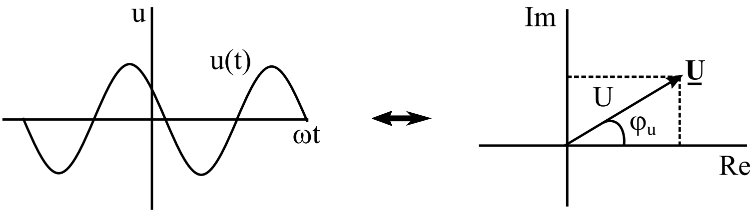
The **phasor representation of a sinusoidal signal** is a complex number that contains information on the rms value and the phase angle of the sinusoidal function.

$$\underline{\mathbf{U}} = U \cdot e^{j\varphi_u} = U\angle\varphi_u$$

Phasor representation of a sinusoidal function

Phasors represent sinusoidal functions in the **frequency domain**

Time domain　　　　　　Frequency domain



$$u(t) = \sqrt{2} \cdot Re(U \cdot e^{j\varphi_u} \cdot e^{j\omega t})$$

$$\underline{\mathbf{U}} = U \cdot e^{j\varphi_u} = U\angle\varphi_u$$

$$u(t) = \sqrt{2} \cdot Re(\underline{\mathbf{U}} \cdot e^{j\omega t})$$

Kirchhoff's laws in phasor form

It can be proved that KCL and KVL are also verified for the phasors associated to voltages and currents.

$$i_k(t) = \sqrt{2} \cdot I_k \cdot \cos(\omega t + \varphi_{i,k}) \qquad \underline{\mathbf{I}_k} = I_k \angle \varphi_{i,k}$$

Kirchhoff's current law:

$$\sum_{k=1}^{n} i_k(t) = 0 \qquad \sum_{k=1}^{n} \underline{\mathbf{I}_k} = 0$$

$$u_k(t) = \sqrt{2} \cdot U_k \cdot \cos(\omega t + \varphi_{u,k}) \qquad \underline{\mathbf{U}_k} = U_k \angle \varphi_{u,k} = 0$$

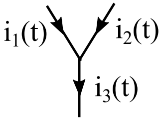
Kirchhoff's voltage law :

$$\sum_{k=1}^{n} u_k(t) \qquad \sum_{k=1}^{n} \underline{\mathbf{U}_k} = 0$$

Example

Calculate $i_3(t)$ given that:

$$i_1(t) = \sqrt{2} \cdot 10 \cdot \cos(25t + 45^o)A \qquad i_2(t) = \sqrt{2} \cdot 20 \cdot \cos(25t + 90^o)A$$



Solution

$$\underline{\mathbf{I}}_1 = 10\angle 45^o A = 7.07 + 7.07jA \qquad \underline{\mathbf{I}}_2 = 20\angle 90^o A = 20jA$$

$$\underline{\mathbf{I}}_3 = \underline{\mathbf{I}}_1 + \underline{\mathbf{I}}_2 = 10\angle 45^o + 20\angle 90^o = 7.07 + 27.07j = 28\angle 75.36^o A$$

$$i_3(t) = \sqrt{2} \cdot 28 \cdot \cos(25t + 75.36^o)A$$

Impedance of a passive element

For capacitors and inductors, the relationships between voltages and currents are given by differential equation.

$$u_L(t) = L \cdot \frac{di_L(t)}{dt} \qquad i_C(t) = C \cdot \frac{du_C(t)}{dt} \qquad u_R(t) = R \cdot i_R(t)$$

To avoid the need of solving differential equations in the analysis of AC circuits we define the **impedance of a passive element**:

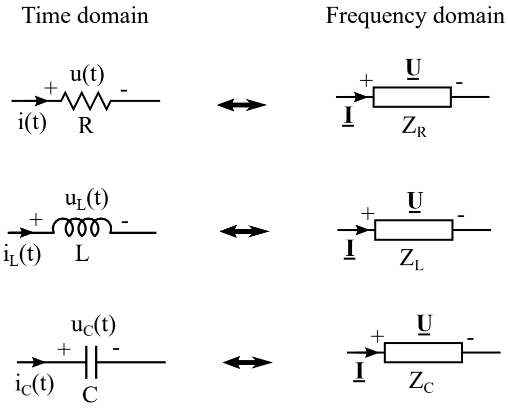
$$Z = \frac{\underline{\mathbf{U}}}{\underline{\mathbf{I}}} \qquad [V] \cdot [A]^{-1} = [\Omega]$$

Admitance: Inverse of the impedance:

$$Y = \frac{1}{Z} \qquad [\Omega]^{-1} = [S]$$

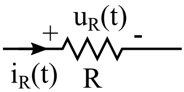
Ohm's law in the frequency domain

We represent the three types of passive elements as impedances.



Ohm's law in the frequency domain: $\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}}$

Impedance of a resistor



$$u_R(t) = R \cdot i_R(t)$$

Time domain:

$$i_R(t) = \sqrt{2} \cdot I_R \cdot \cos(\omega t + \varphi_i)$$

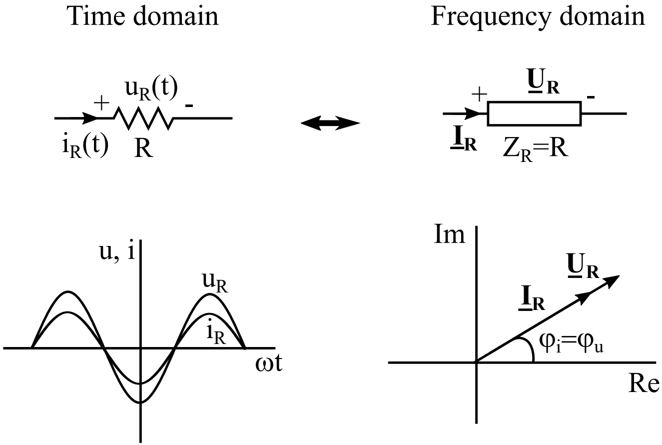
$$u_R(t) = \sqrt{2} \cdot R \cdot I_R \cdot \cos(\omega t + \varphi_i)$$

Frequency domain:

$$\underline{\mathbf{I}}_R = I_R \angle \varphi_i \qquad \underline{\mathbf{U}}_R = R \cdot I_R \angle \varphi_i \qquad Z_R = \frac{\underline{\mathbf{U}}_R}{\underline{\mathbf{I}}_R} = \frac{R \cdot I_R \angle \varphi_i}{I_R \angle \varphi_i} = R$$

$$\boxed{\textbf{Impedance of a resistor:} \quad Z_R = R}$$

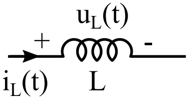
Resistors in the time and frequency domain



Voltages and currents in resistors are in phase:

$$\varphi_i = \varphi_u \qquad \Delta\varphi_{u,i} = 0$$

Impedance of an inductor



$$u_L(t) = L \cdot \frac{di_L(t)}{dt}$$

Time domain:

$$i_L(t) = \sqrt{2} \cdot I_L \cdot \cos(\omega t + \varphi_i)$$

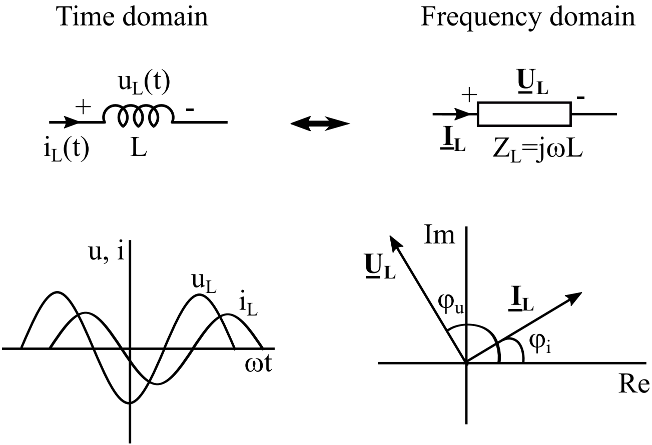
$$u_L(t) = -\sqrt{2} \cdot \omega \cdot L \cdot I_L \cdot sin(\omega t + \varphi_i) = -\sqrt{2} \cdot \omega \cdot L \cdot I_L \cdot \cos(\omega t + \varphi_i - \frac{\pi}{2})$$

Frequency domain:

$$\underline{\mathbf{I}}_L = I_L \angle \varphi_i \quad \underline{\mathbf{U}}_L = \omega \cdot L \cdot I_L \angle \varphi_i + \frac{\pi}{2} \quad Z_L = \frac{\omega \cdot L \cdot I_L \angle \varphi_i + \frac{\pi}{2}}{I_L \angle \varphi_i} = j\omega L$$

Impedance of an inductor: $Z_L = j\omega L$

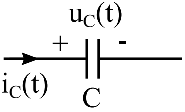
Inductors in the time and frequency domain



In inductors voltage lead currents by 90^o.

$$\varphi_u = \varphi_i + \frac{\pi}{2} \qquad \Delta\varphi_{u,i} = \frac{\pi}{2} = 90^o$$

Impedance of a capacitor



$$i(t) = C \cdot \frac{du(t)}{dt}$$

Time domain:

$$u_C(t) = \sqrt{2} \cdot U_C \cdot \cos(\omega t + \varphi_u)$$

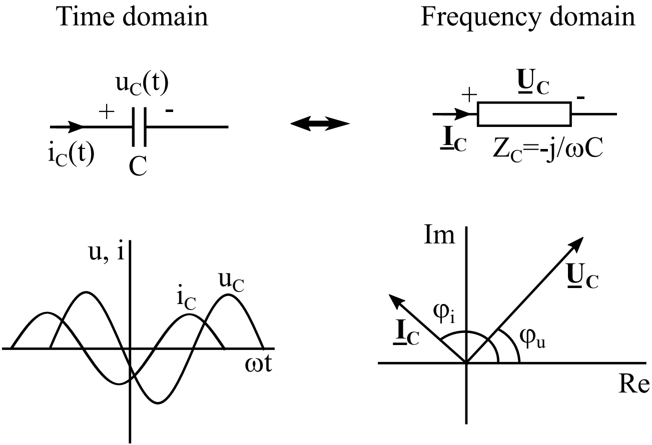
$$i(t) = -\sqrt{2} \cdot \omega \cdot C \cdot U_C \cdot sin(\omega t + \varphi_u) = \sqrt{2} \cdot \omega \cdot C \cdot U_C \cdot \cos(\omega t + \varphi_u - \frac{\pi}{2} + \pi)$$

Frequency domain:

$$\underline{\mathbf{U}}_C = U_C \angle \varphi_u \quad \underline{\mathbf{I}}_C = j\omega C U_C \angle \varphi_u \quad Z_C = \frac{U_C \angle \varphi_u}{j\omega C U_C \angle \varphi_u} = \frac{-j}{\omega C}$$

Impedance of a capacitor: $Z_C = -j/\omega C$

Capacitors in the time and frequency domain



In capacitor current lead voltage by 90^o.

$$\varphi_i = \varphi_u + \frac{\pi}{2} \qquad \Delta\varphi_{u,i} = -\frac{\pi}{2} = -90^o$$

Association of impedances

- Association of passive elements of different nature is not possible in the time domain:

$$u_R(t) = R \cdot i_R(t) \quad u_L(t) = L \cdot \frac{di_L(t)}{dt} \quad i_C(t) = C \cdot \frac{du_C(t)}{dt}$$

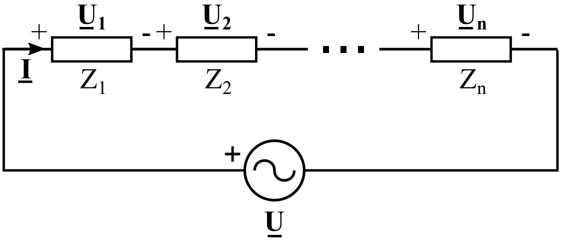
- In the frequency domain the relation between $\underline{\mathbf{U}}$ and $\underline{\mathbf{I}}$ is the same for R, L and C:

$$\underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}}$$

- The representation of R, L and C with impedances makes possible the association of passive elements of different nature.

Association of impedances in series

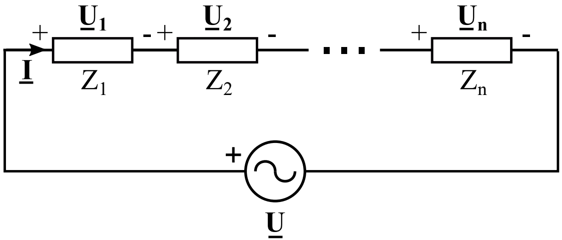
Two or more impedances are series connected if the same phasor current flows through them:



The set of n impedances can be redrawn as an equivalent impedance Z_{eq}

$$Z_{eq} = Z_1 + Z_2 + ... + Z_n = \sum_{k=1}^{n} Z_k$$

Voltage divider equation

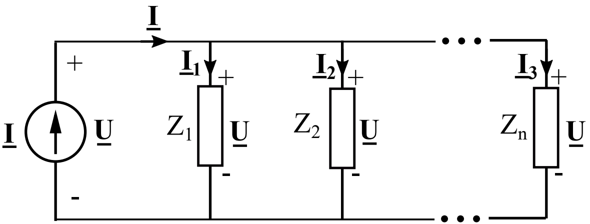


The voltage divider equation is also valid in the frequency domain.
The phasor voltage across impedance k is:

$$\underline{\mathbf{U}}_k = \frac{Z_k}{Z_{eq}} \cdot \underline{\mathbf{U}}$$

Association of impedances in parallel

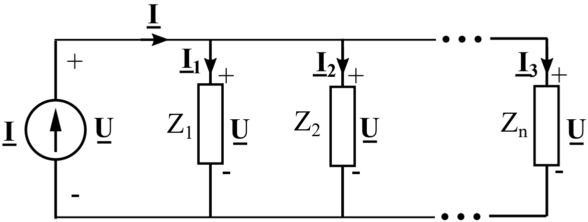
Two or more impedances are parallel connected if they have the same phasor voltage across them:



The set of n impedances can be redrawn as an equivalent impedance Z_{eq}

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + ... + \frac{1}{Z_n} = \sum_{k=1}^{n} \frac{1}{Z_k} \qquad Y_{eq} = \sum_{k=1}^{n} Y_k$$

Current divider equation



The current divider equation is also valid in the frequency domain. The phasor curretn through across impedance k is:

$$\underline{\mathbf{I}}_k = \frac{Y_k}{Y_{eq}} \cdot \underline{\mathbf{I}}$$

* For the case of two impedances in parallel:

$$\underline{\mathbf{I}}_1 = \frac{Y_1}{Y_1 + Y_2} \cdot \underline{\mathbf{I}} = \frac{Z_2}{Z_1 + Z_2} \cdot \underline{\mathbf{I}} \qquad \underline{\mathbf{I}}_2 = \frac{Y_2}{Y_1 + Y_2} \cdot \underline{\mathbf{I}} = \frac{Z_1}{Z_1 + Z_2} \cdot \underline{\mathbf{I}}$$

Complex impedance

- The impedances that represent the three passive elements are:

$$Z_R = R \ \in \mathbb{R} \qquad Z_L = j\omega \cdot L \ \in \mathbb{C} \qquad Z_C = \frac{-j}{\omega \cdot C} \ \in \mathbb{C}$$

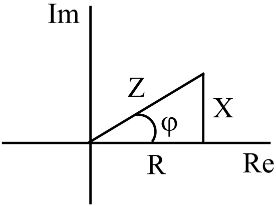
- If we add two or more elements of different nature may find impedances with real and imaginary part:

$$Z = R + jX$$

- The real part of the impedance always comes from resistive elements, and it is called the **Resistance** (R).
- The imaginary part always comes from inductors and capacitors and is called the **Reactance** (X).

Components of a complex impedance

Impedance triangle:



$$Z = R + jX$$

$$\varphi = \arctan \frac{X}{R}$$

The **power factor** of an impedance is the cosine of φ

	Resistance	Reactance	Impedance
Resistor	R	0	R
Inductor	0	$\omega \cdot L$	$j\omega \cdot L$
Capacitor	0	$-1/\omega \cdot C$	$-j/\omega \cdot C$

Mesh current method

1. Assign a **phasor mesh current** to each mesh of the circuit
2. Apply 2KL in phasor form to every mesh of the circuit applying a consistent sign criteria and find a system of equations with the mesh currents as unknowns.

   Mesh equations in matrix form are:

   $$[Z] \cdot [\underline{\mathbf{I}}_{mesh}] = [\underline{\mathbf{U}}_g]$$

   $[Z]$ is the **impedance matrix**:

   Zii=Sum of the impedances in mesh i

   Zij= - Sum of the impedances shared by mesh i and j
3. Solve the equations to find the currents.

Node voltage method

1. Assign a **phasor node voltage** to each node of the circuit and label one of them as **reference node**.
2. Apply KCL in phasor form to each node of the circuit with a consistent sign criteria

   Nodal equations in matrix form are:

   $$[Y] \cdot [\underline{\mathbf{U}}_{node}] = [\underline{\mathbf{I}}_g]$$

   $[Y]$ is the **Admitance matrix** whose terms are:

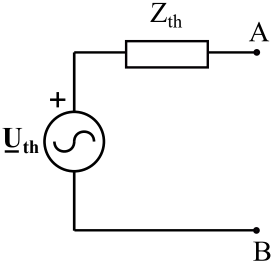
   Yii=Sum of the admitances connected to node i

   Yij= - Sum of the admitances shared by nodes i and j
3. Solve the equations to find the node voltages.

Thevenin equivalent

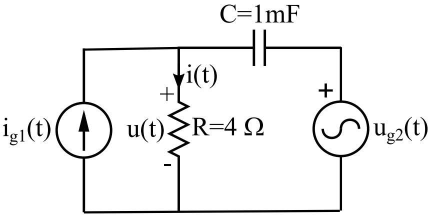
Thenvenin's theorem is valid for AC circuits in the frequency domain.

The Thevenin equivalent of a circuit in the frequency domain consists of a voltage source of value $\underline{\mathbf{U}}_{th}$ and a series impedance Z_{th}.



The methods that can be followed to obtain the parameters of the equivalent are analogous to the ones studied for DC circuits.

Superposition principle for the analysis of circuits with sources of different frequency



$$i_{g1}(t) = \sqrt{2}\cdot 10\cdot\cos(100\cdot t + 90)\,V$$

$$u_{g2}(t) = \sqrt{2}\cdot 50\cdot\cos(200\cdot t - 90)\,V$$

Which frequency should we consider for the calculation of the impedances?

Superposition principle: "The response of a linear circuit subjected to several excitation sources acting simultaneously equals the sum of the responses of the circuits when the sources act separately".

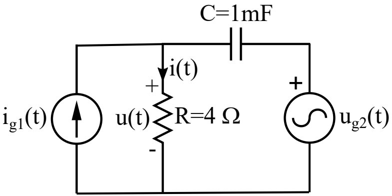
Application of the method: stage 1

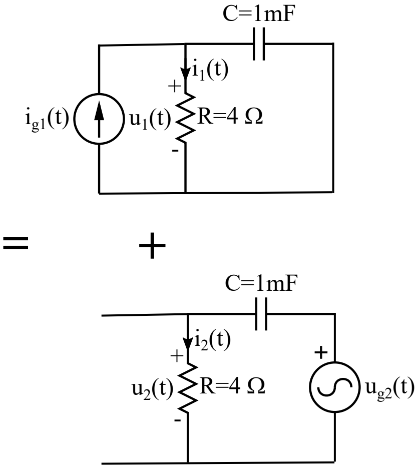
1. Turn off all the sources of frequency ω_2. Replace the voltage sources by short circuits and the current sources by open circuits.
2. Transform the circuit into the frequency domain considering the frequency ω_1 to calculate the impedances of the passive elements.
3. Calculate the response of the circuit in the frequency domain (i.e. the phasors current and voltage at the different parts of the circuit)
4. Find the instantaneous currents and voltages in the time domain which constitute the response of the circuit to the sources of frequency ω_1.

Application of the method: stage 2

1. Turn off all the sources of frequency ω_1.
2. Recalculate the impedances of the passive elements for the frequency ω_2 and obtain the currents and voltages of the elements in the frequency domain
3. Obtain the currents and voltages in the time domain.
4. Obtain the response of the system when all the sources act simultaneously as the sum of the separate responses in the time domain. The currents and voltages will be a sum of two sinusoidal functions of frequencies ω_1 and ω_2.

Example



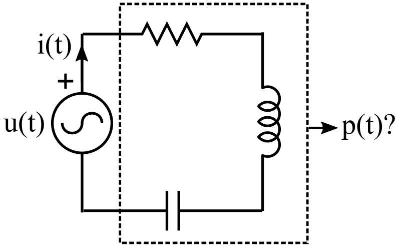


$i=i_1(t)+i_2(t)$

$u=u_1(t)+u_2(t)$

Instantaneous power

Imagine that we want to calculate the power absorbed by the RLC net shown in the diagram:



$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi_u) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t + \varphi_i)$$

The phase angle between u(t) and i(t) is: $\varphi = \varphi_u - \varphi_i$

For simplicity we take the current as phase origin:

$$u(t) = \sqrt{2} \cdot U \cdot \cos(\omega t + \varphi) \qquad i(t) = \sqrt{2} \cdot I \cdot \cos(\omega t)$$

Instantaneous power of an AC circuit

The instantaneous electric power is the product between u(t) and i(t):

$$p(t) = u(t) \cdot i(t) = 2 \cdot U \cdot I \cdot \cos(\omega t + \varphi) \cdot \cos(\omega t)$$

using: $\cos\alpha \cdot \cos\beta = \frac{1}{2} \cdot (\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$$p(t) = U \cdot I \cdot \cos\varphi + U \cdot I \cdot \cos(2\omega t + \varphi)$$

using: $\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$$p(t) = U \cdot I \cdot cos\varphi + U \cdot I \cdot (cos2\omega t \cdot \cos\varphi - \sin 2\omega t \cdot \sin\varphi)$$

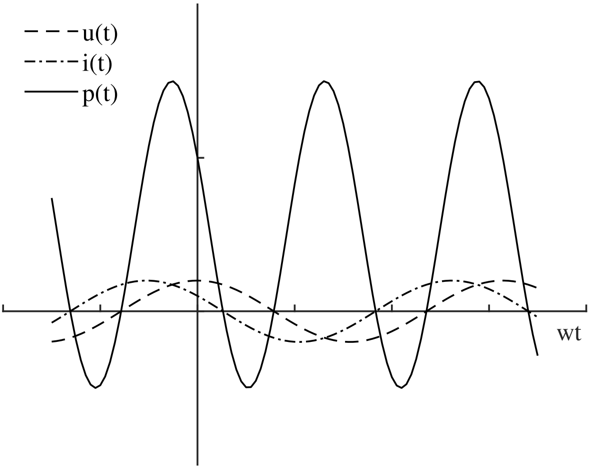
Rearranging:

$$\boxed{\text{p(t)} = \text{U}{\cdot}I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t}$$

Instantaneous power of an AC circuit

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$

p(t) is sometimes + and sometimes -

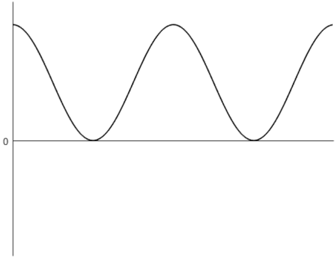


Understanding the instantaneous power

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$

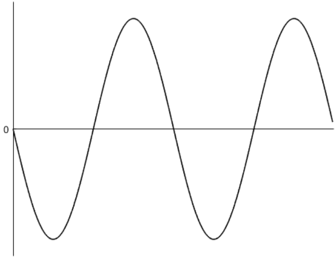
We analyse the two terms of the power separately:

$$U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t)$$



Average value $U \cdot I \cdot cos\varphi$

$$-U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$



Amplitude $U \cdot I \cdot \sin\varphi$

Understanding the instantaneous power

- The instantaneous power fluctuates with frequency 2ω being sometimes positive and sometimes negative
- p(t) is made of two terms:
  1. One is always positive and of average vale $U \cdot I \cdot cos\varphi$.

     This term is the power absorbed in resistors

     $$P = U \cdot I \cdot cos\varphi \qquad \textbf{Active power}$$

  2. The other one is sometimes positive and sometimes negative with average value and amplitude $U \cdot I \cdot sin\varphi$

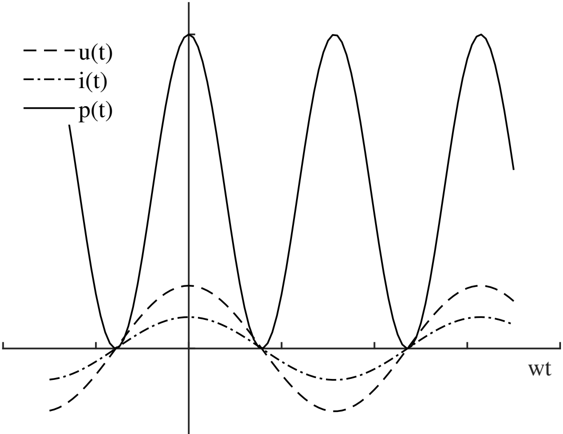
     This term is the power exchanged with the source by inductors and capacitors

     $$Q = U \cdot I \cdot sin\varphi \qquad \textbf{Reactive power}$$

Instantaneous power of a resistor

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$

$$\varphi_R = \varphi_u - \varphi_i = 0 \qquad p_R(t) = U \cdot I \cdot (1 + \cos 2\omega t)$$



Instantaneous power of a resistor

The instantaneous power of a resistor is:

$$p_R(t) = U \cdot I \cdot (1 + \cos 2\omega t)$$

Remarks:

- The power is always positive. This is consistent with what we studied before, because resistors **always** dissipate energy.
- The power fluctuates with frequency 2ω
- The average value of the power is $U \cdot I$

Active and reactive power of resistors

Given the relation between $\underline{\mathbf{U}}$ and $\underline{\mathbf{I}}$ in resistors:

$$\underline{\mathbf{U}} = R \cdot \underline{\mathbf{I}} \qquad \varphi_R = 0^o$$

$$P_R = U \cdot I \cdot \cos\varphi_R = U \cdot I \qquad Q_R = U \cdot I \cdot \sin\varphi_R = 0$$

P can be also expressed in terms of the resistance:

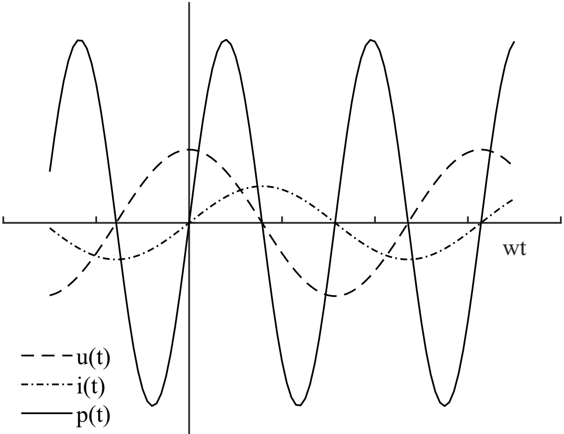
$$U = R \cdot I \qquad P_R = R \cdot I^2 = \frac{U^2}{R} > 0$$

The active power of a resistor is always positive; **resistors always absorb active power**.

Instantaneous power of an inductor

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$

$$\varphi_L = \varphi_u - \varphi_i = 90^o \qquad p_L(t) = -U \cdot I \cdot \sin 2\omega t$$



Instantaneous power of an inductor

The instantaneous power of an inductor is:

$$p_L(t) = -U \cdot I \cdot \sin 2\omega t$$

Remarks:

- The power of an inductor fluctuates with frequency 2ω being sometimes positive and sometimes negative.
- The average value of the instantaneous power is zero.
- These observations are consistent with what we studied about power of inductors: Inductors do not dissipate power, but they store energy in a magnetic field.
- In the first part of the cycle the energy is stored, and in the second part the energy is released and returned to the source.

Active and reactive power of inductors

Given the relation between $\underline{\mathbf{U}}$ and $\underline{\mathbf{I}}$ in an inductor:

$$\underline{\mathbf{U}} = j \cdot \omega \cdot L \cdot \underline{\mathbf{I}} \qquad \varphi_L = 90^o$$

$$P_L = U \cdot I \cdot \cos\varphi_L = 0 \qquad Q_L = U \cdot I \cdot \sin\varphi_L = U \cdot I$$

Q can be also expressed in terms of the reactance:

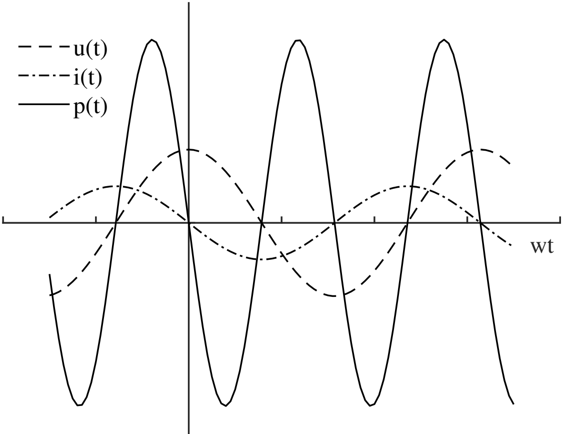
$$U = \omega \cdot L \cdot I = X_L \cdot I \qquad Q_L = X_L \cdot I^2 = \frac{U^2}{X_L} > 0$$

The reactive power of an inductor is always positive; we say that **inductors always absorb reactive power**.

Instantaneous power of a capacitor

$$p(t) = U \cdot I \cdot cos\varphi \cdot (1 + \cos 2\omega t) - U \cdot I \cdot \sin\varphi \cdot \sin 2\omega t$$

$$\varphi_C = \varphi_u - \varphi_i = -90^o \qquad p_C(t) = U \cdot I \cdot \sin 2\omega t$$



Instantaneous power of a capacitor

The instantaneous power of a capacitor is:

$$p_C(t) = U \cdot I \cdot \sin 2\omega t$$

Remarks:

- Power fluctuates with frequency 2ω being sometimes positive and sometimes negative, what means that the capacitor absorbs and delivers power alternatively.
- The average value of the instantaneous power is zero.
- The observations are consistent with the fact that capacitors do not dissipate power, but store energy in an electric field instead.
- In the first part of the cycle the energy is stored and then the energy is released and returned to the source.

Active and reactive power of a capacitor

$$\underline{\mathbf{U}} = \frac{-j}{\omega \cdot C} \cdot \underline{\mathbf{I}} \qquad \varphi_C = -90^o$$

$$P_C = U \cdot I \cdot \cos\varphi_C = 0 \qquad Q_C = U \cdot I \cdot \sin\varphi_C = -U \cdot I$$

Q can be also expressed in terms of the reactance:

$$U = \frac{1}{\omega \cdot C} \cdot I = -X_C \cdot I \qquad Q_C = X_C \cdot I^2 = \frac{U^2}{X_C} < 0$$

The reactive power of a capacitor is always negative; we say that **capacitors always deliver reactive power**.

Power absorbed vs. power exchanged

Two different behaviours in relation to energy in AC circuits:

1. Resistors absorb power from the sources and transform it into heat.
2. Inductors and capacitors absorb power from the sources, store it and return it later.

The absorbed power and the fluctuating power are both important and must be taken into account in the analysis of AC circuits.

Nevertheless they have a different impact on circuits because the first supposes a real consumption of energy the second is a continuous flow of energy between the source and the inductive and capacitive loads.

Active and reactive power

Two types of power in AC circuits:

- **Active power (P)**: Power absorbed in resistors.

$$P = U \cdot I \cdot \cos\varphi$$

  Active power is measured in Watts [W]

- **Reactive power (Q)**: is the power that fluctuates between the inductors and capacitors and the sources.

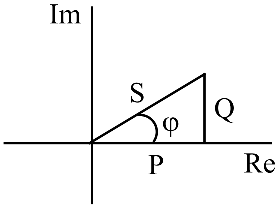
$$Q = U \cdot I \cdot \sin\varphi$$

  Reactive power is measured in Volt-Ampere reactive [var]

- **Complex power ($\mathcal{S}$)**:

$$\mathcal{S} = P + Qj \qquad [VA]$$

Power triangle and other power-related parameters



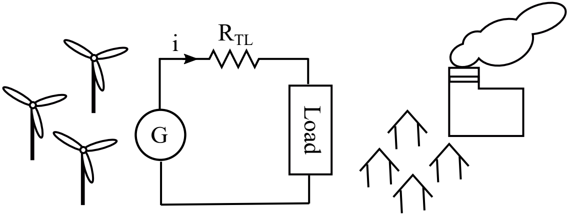
Apparent power (S): Is the modulus of the complex power:

$$S = \sqrt{P^2 + Q^2} = U \cdot I \qquad [VA]$$

Power factor: cosine of the phase shift between voltage and current.

$$p.f. = \cos\varphi = \cos(\varphi_u - \varphi_i)$$

Practical importance of the power factor



- Inductors and capacitors do not absorb power, but they exchange it with the source
- However the flow of energy overloads the lines and produces losses
- Electric companies charge not for the active power but for the apparent power consumed.
- Ideally the power factor must be 1 (Q=0)

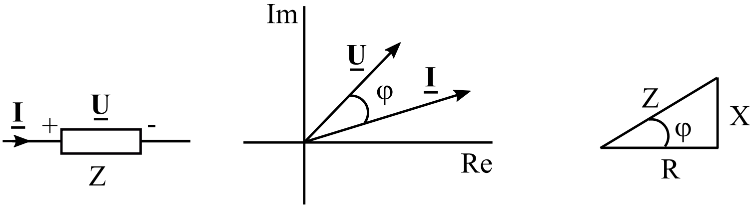
Link

Active power and reactive power of resistors inductors and capacitors

Resistor	$\varphi_R = 0^o$	$P_R = R \cdot I^2$	$Q_R = 0$
Inductor	$\varphi = 90^o$	$P_L = 0$	$Q_L = X_L \cdot I^2 = \omega \cdot L \cdot I^2 > 0$
Capacitor	$\varphi = -90^o$	$P_C = 0$	$Q_C = X_C \cdot I^2 = -\frac{1}{\omega \cdot C} \cdot I^2 < 0$

Resistors absorb active power, inductors absorb reactive power and capacitors deliver reactive power

Power of a complex impedance



$$Z = R + jX \qquad \underline{\mathbf{U}} = Z \cdot \underline{\mathbf{I}}$$

Active and reactive power

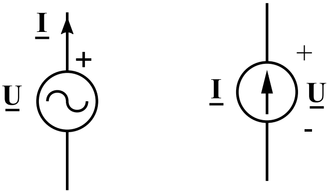
$$P = U \cdot I \cdot \cos\varphi = R \cdot I^2 \qquad Q = U \cdot I \cdot \sin\varphi = X \cdot I^2$$

Complex power

$$\mathcal{S} = P + Qj = Z \cdot I^2$$

Complex power of AC souces

Sign criteria for power: A source delivers power when current flows from the terminal at lower voltage towards the terminal at higher voltage. Power delivered by sources is taken as positive.



$$\mathcal{S}_g = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* = P_g + Q_g j$$

The real part of the complex power is the active power delivered by the source and the imaginary part is the reactive power delivered by the source. In some cases P_g or Q_g might be negative, what means that the source absorbs active or reactive power.

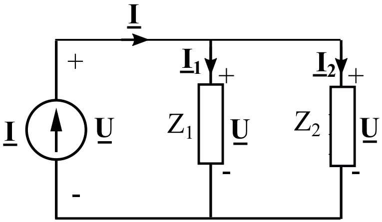
Boucherot theorem: power balance in AC circuits

The total amount of active and reactive power absorbed in an electric circuit equals the sum of the active and reactive power absorbed by its passive elements.

$$P_T = \sum_k P_k \qquad\qquad Q_T = \sum_k Q_k$$

This implies that the sum of the complex power supplied by the sources equals the sum of the complex power absorbed by the impedances.

Example



The power supplied by the source equals the sum of the complex power absorbed by the impedances Z_1 and Z_2.

$$\mathcal{S}_g = \underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^* = \underline{\mathbf{U}} \cdot (\underline{\mathbf{I}}_1 + \underline{\mathbf{I}}_2)^* = S_{Z_1} + S_{Z_2} = (P_1 + P_2) + j(Q_1 + Q_2)$$