Module 2: Analysis of DC circuits

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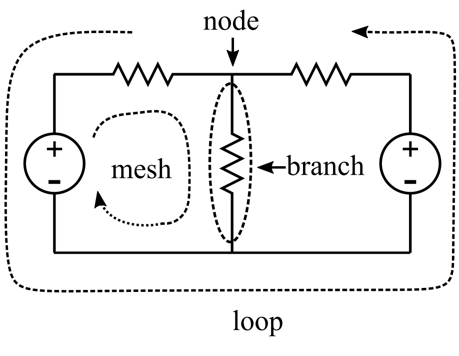
In this module we will study different methods that can be applied to the systematic analysis of electrical circuits and we will apply them to the analysis of circuits supplied in Direct Current (DC). The same methods will be applied to the analysis of Alternating Current (AC) circuits in Module 3.

1 Definitions

Prior to introducing the methods we need to define some terms related with the topology of electrical circuits:

- **Branch**: Part of the circuit with two terminals
- **Node**: Junction of two or more branches
- **Loop**: Closed path in a circuit
- **Mesh**: Closed path in a circuit that does not have any other closed path inside it



2 Mesh current method

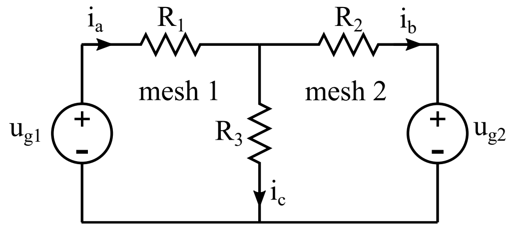
Mesh current method (also referred to as mesh analysis) is a systematic method based in the application of Kirchhoff's voltage law (KVL) to every mesh of a circuit.

The method can be applied to the analysis of any planar circuit [1].

[1] Planar circuits are the ones that can be drawn in a plane with no crossing branches

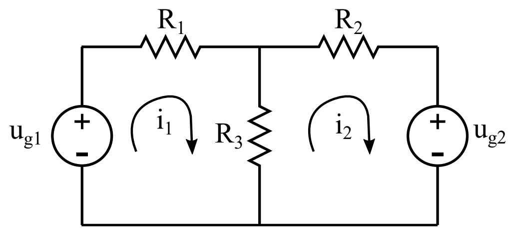
2.1 Application of mesh current method

Imagine that we want to solve the circuit of the figure. The circuit has two mesh: mesh 1 at the left and mesh 2 at the right. Note that R_1 is part of mesh 1 and R_2 is part of mesh 2, while R_3 is shared by both mesh.



To calculate the branch currents of the circuit (ia, ib and ic) using mesh current method the steps below must be followed:

1. Define a "mesh current" for each mesh of the circuit. We consider that all the mesh currents flow in the same direction [2]



2. Apply KVL to every mesh of the circuit considering that the current that flows through each mesh is the correspondent mesh current.

$$\sum_k u_k = 0$$

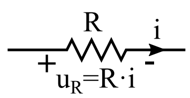
Note that:

- A sign criteria must be adopted to apply KVL. Any criteria is valid if we are consistent with it. In this notes we consider:

Voltage drops +

Voltage rises -

- As resistors are passive elements we consider that there is always a voltage drop across them. The value of the voltage drop can be calculated according to Ohm's law:



[2]It is valid to define the mesh currents with different directions. However, for the sake of simplicity clockwise direction will be assigned to all the mesh currents in these notes.

3. We obtain a system of equations with as many equations as mesh in the circuit:

   Equation for mesh 1: $-u_{g1} + R_1 \cdot i_1 + R_3 \cdot (i_1 - i_2) = 0$

   Equation for mesh 2: $R_2 \cdot i_2 + R_3 \cdot (i_2 - i_1) + u_{g2} = 0$

4. We solve the system of equations and find the values for mesh currents i_1 and i_2.

   It is useful to write mesh equations in matrix form to solve the system more easily. This is specially interesting for circuits with three or more mesh.

   The mesh equations in matrix form are:

$$\begin{pmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{pmatrix} \cdot \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} u_{g1} \\ -u_{g2} \end{pmatrix}$$

   These equations can be generalized as:

$$\left[\mathfrak{R}\right] \cdot \left[I_{mesh}\right] = \left[U_g\right]$$

   where $\left[\mathfrak{R}\right]$ is the "resistance matrix" whose terms are:

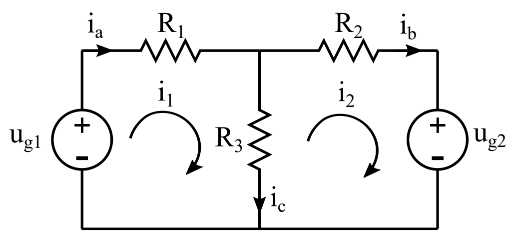
   $R_{jj} =$ Total resistance in mesh j

   $R_{jk} = -$ Total resistance shared by mesh j and mesh k

   The vector $\left[I_{mesh}\right]$ contains the mesh currents and the terms of the vector $\left[Ug\right]$ are the voltage rises produced by sources across every mesh:

   $Ug_i = \sum$ voltage rises across voltage sources in mesh i

5. Finally we calculate the branch currents or other requested variables (power, voltages..), using the values of the mesh currents.



$$i_a = i_1$$

$$i_b = i_2$$

$$i_c = i_1 - i_2$$

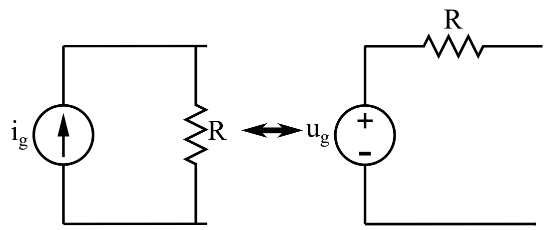
2.2 Application of mesh current method in circuits with current sources

The application of mesh current method to the analysis of circuits that contain resistors and voltage sources is quite straightforward; the voltage drop across resistors is defined by Ohm's law and the voltage drop across voltage sources is given by the output voltage of the source.

The case of circuits that incorporate current sources is more challenging because current sources provide a certain value of current, but the voltage drop across them depends on the circuit configuration and then it is not so immediate to include these elements in mesh equations. Depending on the configuration of the current sources different approaches can be considered, as is explained in this section.

2.2.1 Real current sources

As was studied in module 1, real current sources can always be redrawn as equivalent real voltage sources according to the following transformation rule:



$$u_g = i_g \cdot R$$

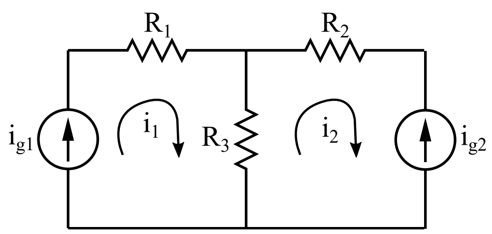
In circuits with real current sources, the next steps might be followed to facilitate the application of mesh current method:

1. Redraw the real current sources as real voltage sources.
2. Solve the circuit applying mesh current method.
3. Go back to the original circuit and use the obtained mesh currents to find the requested variables (i.e. voltages, currents, powers..).

2.2.2 Ideal current sources that belong to one mesh

If we want to solve a circuit that contains an ideal current source in a branch that belongs exclusively to one mesh, we can just assume that the mesh current equals the current of the current source.

In the following circuit:



The mesh equations can be written as:

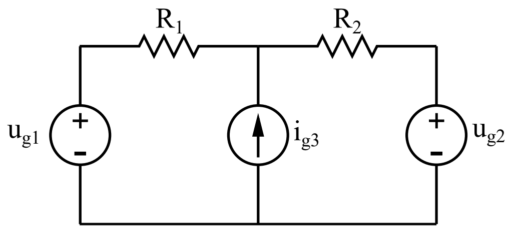
Mesh 1: $i_1 = i_{g1}$

Mesh 2: $i_2 = -i_{g2}$

Note that the resistors R_1 and R_2 that are in series with the ideal current sources have no effect on the current flowing through the branches.

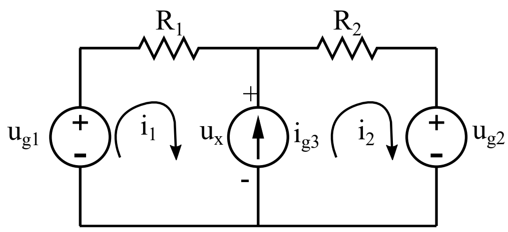
2.2.3 Ideal current sources that belong to more than one mesh

There are other cases in which the circuit contains an ideal current source that belongs to more than one mesh:



In this case we can not redraw the current source as a voltage source and neither can identify the current of the source with any of the mesh current. However, two different strategies might be applied to find the mesh equations:

1. Define a new unknown "u_x",which is the voltage drop across the ideal current source.



   This new unknown is included in the equations. The system must be completed with an additional equation that relates the current of the ideal current source with the mesh currents:

   Mesh 1: $-u_{g1} + R_1 \cdot i_1 + u_x = 0$

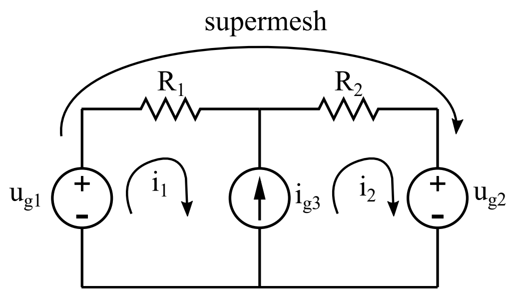
   Mesh 2: $R_2 \cdot i_2 - u_x + u_{g2} = 0$

   Additional equation: $i_{g3} = i_2 - i_1$

   These equations might be expressed in matrix form as well as:

$$\begin{pmatrix} R_1 & 0 & 1 \\ 0 & R_2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} i_1 \\ i_2 \\ u_x \end{pmatrix} = \begin{pmatrix} u_{g1} \\ -u_{g2} \\ i_{g3} \end{pmatrix}$$

2. As an alternative to the previous approach, the **supermesh** method can be applied. A supermesh is a closed path that contains the branch where the ideal current source is connected. In the previous example, we might take the external part of the circuit as supermesh.



Applying KVL to the supermesh:

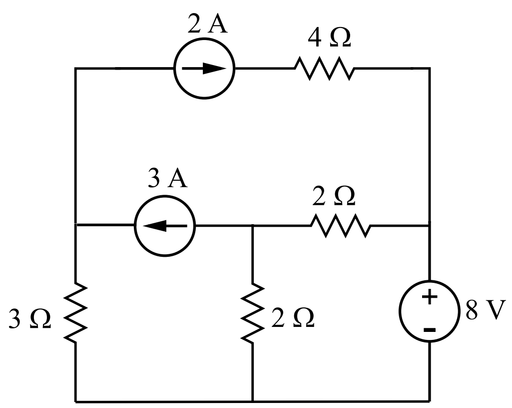
Supermesh equation: $-u_{g1} + R_1 \cdot i_1 + R_2 \cdot i_2 + u_{g2} = 0$

The system of equations must be completed with an additional equation that relates the mesh currents with the current of the ideal current source.

Additional equation: $i_{g3} = i_2 - i_1$

2.3 Example

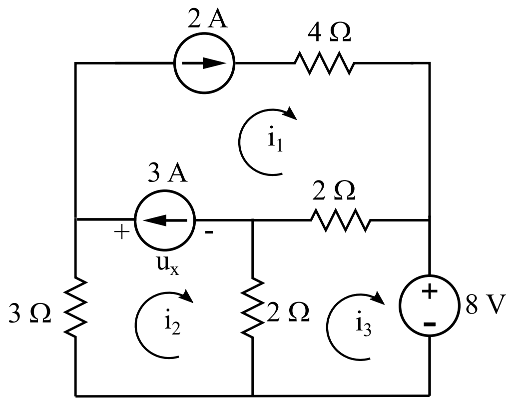
Solve the circuit using mesh analysis and do a power balance.



Solution

The problem is solved using mesh analysis method, although there are other methods and simplifications that could lead to the same solution in a faster way.

Firstly a mesh current is assigned to each mesh of the circuit and the voltage u_x is defined:



The mesh equations are:

Mesh 1: $i_1 = 2$

Mesh 2: $3 \cdot i_2 + u_x + 2 \cdot (i_2 - i_3) = 0$

Mesh 3: $2 \cdot (i_3 - i_2) + 2 \cdot (i_3 - i_1) + 8 = 0$

Additional equation: $i_1 - i_2 = 3$

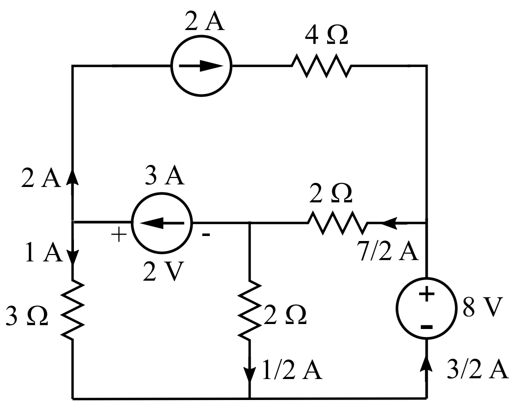
We could write these equations in matrix form or solve the system using any other chosen method:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & 1 \\ -2 & -2 & 4 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ u_x \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -8 \\ 3 \end{pmatrix}$$

Solving the equations we find that

$i_1 = 2A \quad i_2 = -1A \quad i_3 = -3/2A \quad u_x = 2V$

The currents flowing through all the branches of the circuit are obtained:



And calculate the power balance.

Power **absorbed** by resistors:

$p_R = \sum_k R_k \cdot i_k = 4 \cdot 2^2 + 3 \cdot 1^2 + 2 \cdot (7/2)^2 + 2 \cdot (1/2)^2 = 44W$

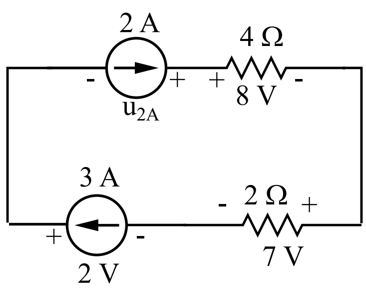
Power **delivered** by sources:

$pg = \sum_k u_k \cdot i_k$

$p_{3A} = 2 \cdot 3 = 6W$

$p_{8V} = 8 \cdot 3/2 = 12W$

The voltage drop across the current source of 2A is calculated applying KVL to a closed path that contains the source:



$-u_{2A} + 8 + 7 - 2 = 0 \qquad u_{2A} = 13V$

$p_{2A} = 2 \cdot 13 = 26W$

$p_{sources} = 26 + 6 + 12 = 44W$

Then, the power balance is verified:

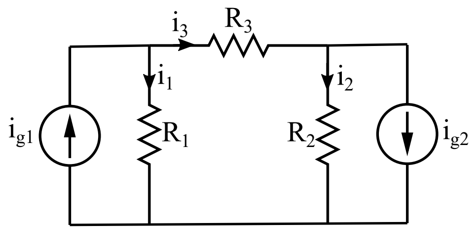
$$\text{Power generated by sources} = \text{Power absorbed by resistors}$$

3 Node voltage method

Node voltage method (also referred to as nodal analysis) is a systematic method based in the application of Kirchoff's current law (KCL) to all the nodes of a circuit.

3.1 Application of node voltage method

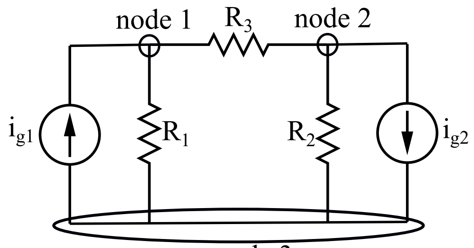
Imagine that we want to solve the circuit below and find the branch currents i_1, i_2 and i_3:



The application of node voltage method consists of the following steps:

1. Firstly we identify the independent nodes of the circuit, i.e. the nodes that have a different voltage level.

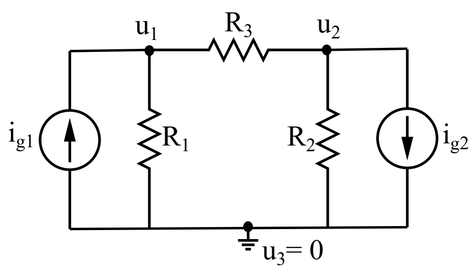
   The circuit of the figure has three independent nodes. All the branches at the bottom of the circuit have the same voltage level, so that part of the circuit is taken as one node. We label the nodes with numbers 1, 2 and 3.



   The node voltage method consists of finding the voltage of each node with respect to a "reference node", whose voltage level is zero.

   In the previous example we assign nodal voltages u_1, u_2 and u_3 to the three nodes and take node 3 as reference node ($u_3 = 0$).

   The objective is to find the voltage of each non reference node with respect the reference node. [3]



2. We apply KCL to each node of the circuit considering that the voltage of each node is the correspondent node voltage:

$$\sum_k i_k = 0$$

   Note that:

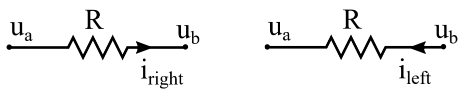
[3] In some problems the reference node is specified while in other cases we will need to assign it. Any node can be taken as reference node. The final solution of the problem is the same for any choice.

- We should adopt a sign criteria to apply 1KL. Any criteria is valid if we are consistent with it. In this notes we will consider that the currents that flow out of a node are positive and the currents that flow into a node are negative.

Current flowing out of a node +

Current flowing into a node -

- The currents flowing through a resistor can be calculated using Ohms's law if we know the voltage drop across it. Then, we calculate the current flowing through each resistor of the circuit as a function of the nodal voltages at their two terminals. The current can be considered to be flowing towards the right or towards the left by changing its sign.



$$i_{right} = \frac{u_a - u_b}{R} \qquad i_{left} = \frac{u_b - u_a}{R}$$

3. We obtain a system of equations with as many equations as nodes in the circuit

$$\textbf{Equation for node 1:} -i_{g1} + \frac{u_1}{R_1} + \frac{u_1 - u_2}{R_3} = 0$$

$$\textbf{Equation for node 2:}\ i_{g1} + \frac{u_2}{R_2} + \frac{u_2 - u_1}{R_3} = 0$$

4. We solve the system and find the values for the node voltages. Sometimes it is useful to express nodal equations in matrix form to solve them more easily. This is specially interesting for circuits with three or more non-reference nodes.

   The equations in matrix notation for the example are:

$$\begin{pmatrix} \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} i_{g1} \\ -i_{g2} \end{pmatrix}$$

   Solving these equations we find the values of u_1 and u_2

   The nodal equations in matrix form can be generalized as:

$$\left[\mathfrak{G}\right] \cdot \left[U_{node}\right] = \left[I_g\right]$$

   where $\mathfrak{G}$ is the so called "conductance matrix"

$$\left[\mathfrak{G}\right] = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

The terms of the main diagonal of the conductance matrix are the self conductances (i.e. total conductance connected to the analysed node) and the non diagonal terms are the shared conductances (i.e. conductance shared by two nodes) with negative sign.

$G_{11} =$ Total conductance connected to node 1

$G_{22} =$ Total conductance connected to node 2

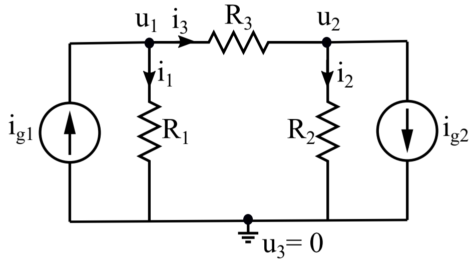
$G_{12} = G_{21} = -$ Total conductance connected between node 1 and node 2

The vectors $\left[U_{node}\right]$ and $\left[Ig\right]$ are:

$$\left[U_{node}\right] = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\left[Ig\right] = \begin{bmatrix} \sum \text{Current injected by current sources into node 1} \\ \sum \text{Current injected by current sources into node 2} \end{bmatrix}$$

5. Finally, the branch currents i_1, i_2 and i_3 are calculated using the node voltages:



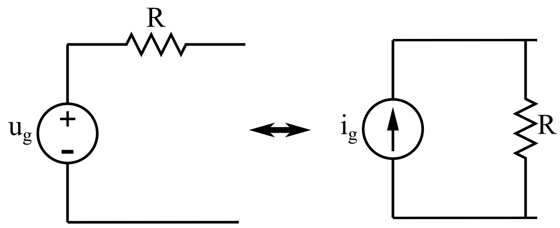
$$i_1 = \frac{u_1}{R_1} \qquad i_2 = \frac{u_2}{R_2} \qquad i_3 = \frac{u_1 - u_2}{R_3}$$

3.2 Application of nodal analysis in circuits with voltage sources

The presence of voltage sources introduces a difficulty for the application of node voltage method to the analysis of a circuit. Voltage sources have a well defined voltage drop but the current flowing through them depends on the configuration of the circuit. In this section several procedures are explained to apply nodal analysis to circuits that incorporate voltage sources.

3.2.1 Real voltage sources

As was studied in module 1, a real voltage source can always be redrawn as an equivalent real current source according to the following transformation rule:



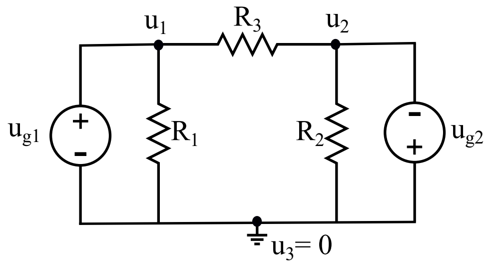
$$i_g = \frac{u_g}{R}$$

Then, to solve circuits that include real voltage sources using nodal analysis, the next steps may be followed:

1. Redraw the real voltage sources as real current sources
2. Solve the circuit applying node voltage method as explained before
3. Go back to the original circuit and use the information obtained from the nodal analysis to find the wanted variables (i.e. voltages, currents, powers..)

3.2.2 Ideal voltage sources connected to the reference node

If we have an ideal voltage source in a circuit which is connected between a non-reference node and the reference node, as the case shown in the drawing below, we can assume that the node voltage equals the voltage of the voltage source.



In that case the nodal equations in this case may be written as:

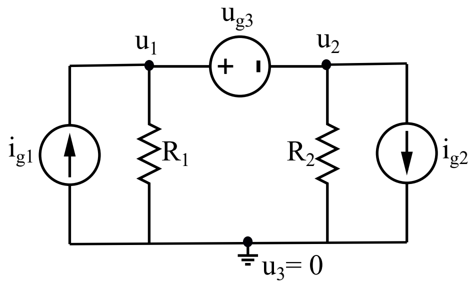
Node 1: $u_1 = u_{g1}$

Node 2: $u_2 = -u_{g2}$

Note that the resistors R_1 and R_2, which are in parallel with the ideal voltage sources, have no effect on the voltage drop across the nodes.

3.2.3 Ideal voltage sources connected between two non-reference nodes

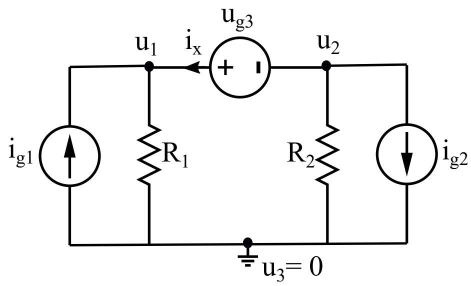
There are other cases in which the ideal voltage source is connected between two non-reference nodes, as in the following circuit:



In this case we can not redraw the voltage source as a current source and can not identify the voltage of the source with any of the nodal voltages either.

Two different strategies can be applied to solve the circuit with node voltage method:

1. Define a new unknown current i_x, which is the current flowing through the ideal voltage source.



   This new unknown will be included in the equations. As we have three unknowns now we need an additional equation to find the node voltages. The additional equation relates the voltage of the ideal voltage source with the node voltages.

   The nodal equations for this case are:

$$\text{Node 1: } -i_{g1} + \frac{u_1}{R_1} - i_x = 0$$

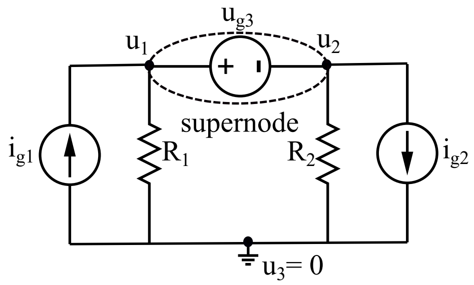
$$\text{Node 2: } i_{g2} + \frac{u_2}{R_2} + i_x = 0$$

$$\text{Additional equation: } u_1 - u_2 = u_{g3}$$

   These equations can be expressed in matrix form as well:

$$\begin{pmatrix} \frac{1}{R_1} & 0 & -1 \\ 0 & \frac{1}{R_2} & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ i_x \end{pmatrix} = \begin{pmatrix} i_{g1} \\ -i_{g2} \\ u_{g3} \end{pmatrix}$$

2. As alternative to the previous method we can write a "supernode" equation. The supernode is a fictional node that contains the ideal voltage source. The supernode is signaled with a discontinuous line in the diagram below:



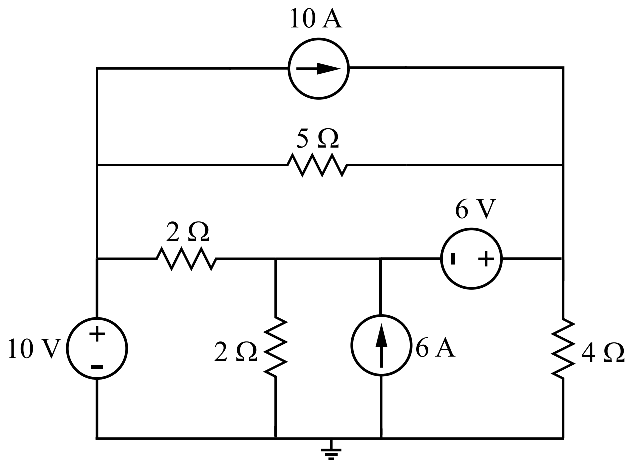
The system must be completed with an additional equation that relates the node voltages with the voltage of the ideal voltage source.

$$\text{Supernode equation:} \; -u_{g1} + \frac{u_1}{R_1} + \frac{u_2}{R_2} + u_{g2} = 0$$

$$\text{Additional equation:} \; u_{g3} = u_1 - u_2$$

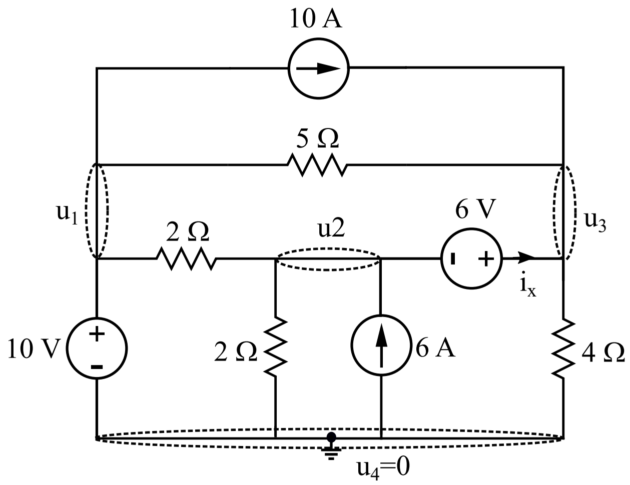
3.3 Example

Solve the following circuit using node voltage method



Solution

Firstly we have to identify the independent nodes and assign them a node voltage. We set the reference node in node 4.



We apply KCL to nodes 1, 2 and 3 and write the nodal equations:

$$\text{Node 1: } u_1 = 10$$

$$\text{Node 2: } \frac{u_2 - u_1}{2} + \frac{u_2}{2} - 6 + i_x = 0$$

$$\text{Node 3: } -10 + \frac{u_3 - u_1}{5} + \frac{u_3}{4} - i_x = 0$$

$$\text{Additional equation: } u_3 - u_2 = 6$$

We can solve the system using matrix algebra or any other chosen method. The system in matrix form would be:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 1 \\ -\frac{1}{5} & 0 & \frac{1}{5} + \frac{1}{4} & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ i_x \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ 10 \\ 6 \end{pmatrix}$$

Solving the equations we find:

$$u_1 = 10V \quad u_2 = 14V \quad u_3 = 20V \quad i_x = -3A$$

We could have also written the nodal equations using the supernode method. Considering a supernode that contains nodes 2 and 3 we find the following system of equations:

$$\text{Node 1: } u_1 = 10$$

$$\text{Supernode: } \frac{u_2 - u_1}{2} + \frac{u_2}{2} - 6 - 10 + \frac{u_3 - u_1}{5} + \frac{u_3}{4} = 0$$

$$\text{Additional equation: } u_3 - u_2 = 6$$

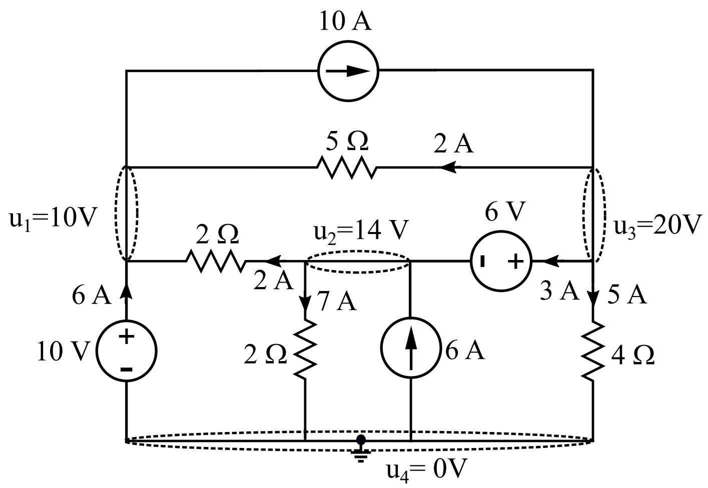
The solution of that system leads to the same solution:

$$u_1 = 10V \quad u_2 = 14V \quad u_3 = 20V$$

We can now obtain the current in every branch of the circuit considering that the current flows from the node with highest voltage to the node of lower voltage and that the value of the current is calculated as:

$$i = \frac{u_{high} - u_{low}}{R}$$

Additionally the currents flowing through voltage sources can be calculated applying KCL to the nodes where they are connected:



Finally we calculate the power balance:

- Power **absorbed** by resistors: $p_R = \sum_k R_k \cdot i_k = 5 \cdot 2^2 + 2 \cdot 2^2 + 2 \cdot 7^2 + 4 \cdot 5^2 = 226W$

- Power **delivered** by sources:

  $p_g = \sum_k u_k \cdot i_k$ [4]

  $p_{10A} = 10 \cdot 10 = 100W$

  $p_{6V} = -6 \cdot 3 = -18W$

  $p_{6A} = 14 \cdot 6 = 84W$

[4] Sources absorb power if the relative polarity of the voltage and current is such that current flows from + to -. In that case the power of the source will be taken as negative. Note that that is the case for the 6V voltage source in this example.

$p_{10V} = -10 \cdot 6 = -60W$

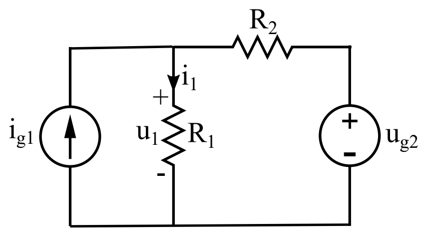
$p_{sources} = 100 - 18 + 84 + 60 = 226W$

4 Superposition principle

The superposition principle states that "the response of a linear circuit subjected to several excitation sources acting simultaneously equals the sum of the responses of the circuits when the sources act separately".

A circuit is linear if the relation between the voltages and currents in all their elements verify a linear relation, which is the case of all the circuits that will be studied in this course.

Superposition principle can be applied to calculate the current i_1 and the voltage u_1 in the circuit below:



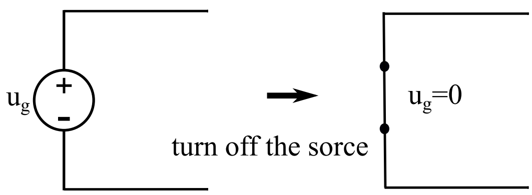
The current i_1 is the sum of the current that flows through the resistor if only the current source is working and the current that flows though it if only the voltage source is working.

The same statement is valid for the calculation of voltage u_1.

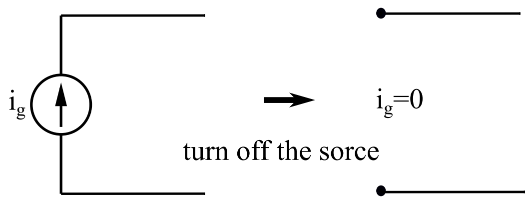
To apply the superposition principle we "turn off" the sources one by one and calculate the response of the circuit to each source. Then we obtain the whole response of the circuit as a sum of both responses.

Note that:

- Cancelling a voltage source consists of considering that its voltage drop is zero. This is equivalent to redrawing it as a short-circuit.

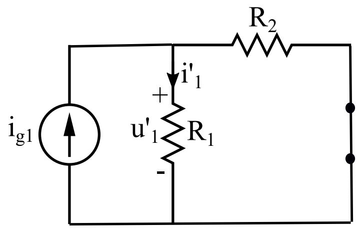


- Cancelling a current source consists of considering that the current flowing through it is zero. This is equivalent to transform the source into open-circuit.



Then, to solve the circuit in the figure using the superposition principle:

1. We "turn off" the voltage source and analyse the response of the circuit when only the current source acts as excitation:



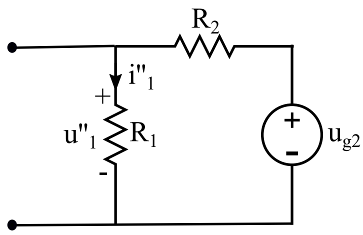
We can calculate the current i_1' using the current divider (note that the two resistors are now in parallel) or any other method that we choose:

$$i_1' = \frac{R_2}{R_1 + R_2} \cdot i_{g1}$$

The voltage u_1' is:

$$u_1' = \frac{R_1 \cdot R_2}{R_1 + R_2} \cdot i_{g1}$$

2. We "turn off" the current source and analyse the response of the circuit when only the voltage source acts as excitation:



In this case, the two resistors are in series and the current i_1'' is:

$$i_1'' = \frac{u_{g2}}{R_1 + R_2}$$

and the voltage u_1'' is:

$$u_1'' = \frac{R_1}{R_1 + R_2} \cdot u_{g2}$$

3. Finally, we find the current i_1 and the voltage u_1 as the sum of the separate responses:

$$i_1 = i_1' + i_1''$$

$$u_1 = u_1' + u_1''$$

Two last remarks about the superposition principle:

- The superposition principle can not be used to calculate the power absorbed by an element of the circuit. If we want to calculate the power absorbed by a resistor (i.e. p_{R_1}) we should remember that:

  $$p_{R_1} \neq p'_{R_1} + p''_{R_1}.$$

  However, we could use the superposition principle to calculate the current i_1 and then use it to obtain the power:

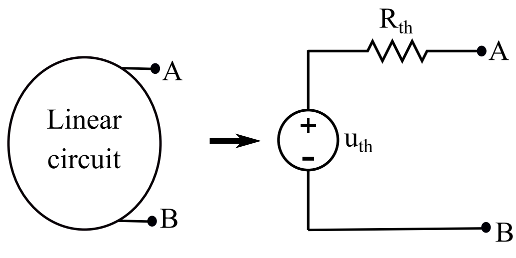
  $$p_{R_1} = R_1 \cdot i_1^2$$

- As will be seen later in the course, the superposition principle is very important when we want to solve an AC circuit that incorporates sources of different frequencies.

5 Application of Thevenin's theorem for circuit analysis

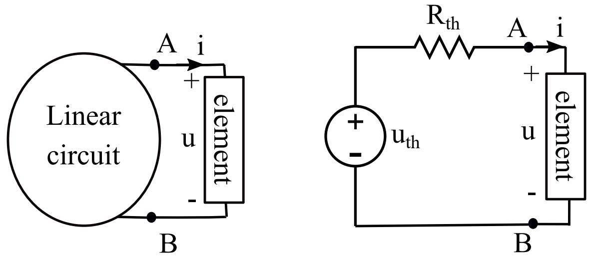
5.1 Thevenin's theorem

Any linear circuit seen from two terminals can be replaced by a simplified circuit consisting of an ideal voltage source in series with a resistor. This simplified circuit is the "Thevenin equivalent of the circuit".



Thevenin equivalent

This means that any element (a resistor, a voltage source, a current source...anything!) connected between A and B would have identical behaviour (i.e. same current flow and same voltage drop across it) if the initial circuit is considered or if the equivalent circuit is considered. It would not be possible to distinguish between the configuration at the left side of the figure below and the configuration at the right side of it:

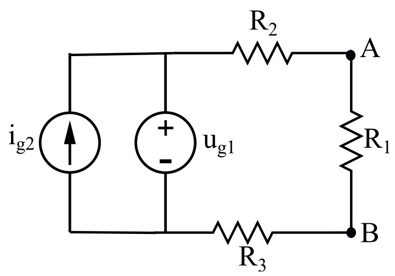


Thevenin's theorem has many practical application such as electronic circuits design or power system analysis.

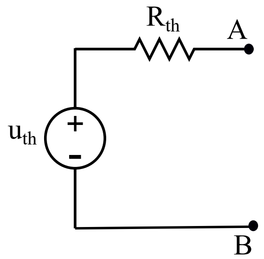
To apply Thevenin's theorem we need to learn how to calculate the two parameters of the equivalent circuit: u_{th} and R_{th}.

5.2 Determination of the Thevenin equivalent of a circuit

We want to find the Thevenin equivalent of the following circuit:



Remember that calculating the Thevenin equivalent of the circuit means determining what couple of values u_{th} and R_{th} make the behaviour of the simplified circuit represented below identical to the one of the initial circuit, from the point of view of the couple of terminals A B.



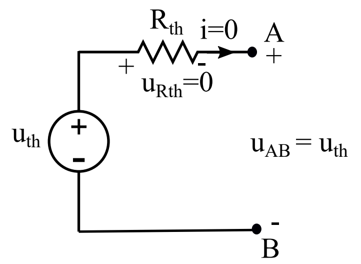
The original circuit and the Thevenin equivalent are equivalent if any element connected between A and B behaves in the same way if we consider the original circuit or the equivalent circuit.

To calculate u_{th} and R_{th} we evaluate how both circuits behave under two particular situations:

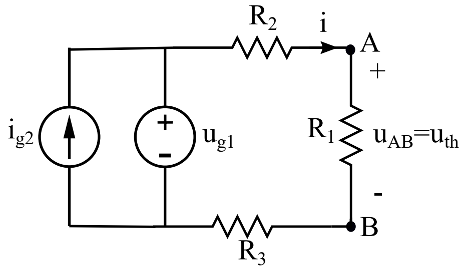
1. The behaviour of the two circuits when we leave A B in open circuit (this is the same than connecting a resistance of value infinite between A and B)
2. The behaviour of the two circuits when we place a short circuit between A and B (this is the same than connecting a resistor of value zero)

5.2.1 Calculation of the Thevenin voltage ($\mathbf{u_{th}}$): open circuit analysis

Firstly, we analyse the behaviour of the equivalent circuit when the circuit is open. In that case the current flowing through between A and B is null and the voltage drop across R_{th} is also zero. As can be seen, under this condition $u_{AB} = u_{th}$



Then, the analysis of the original circuit without connecting anything between A and B provides the value of u_{th}



The current i is:

$$i = \frac{u_{g1}}{R_1 + R_2 + R_3}$$

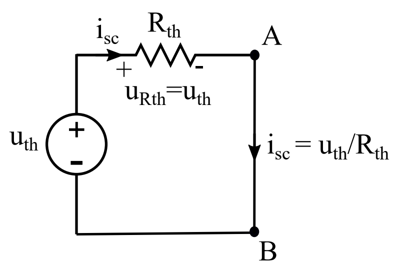
and the voltage between A and B:

$$u_{AB} = R_1 \cdot i = \frac{R_1 \cdot u_{g1}}{R_1 + R_2 + R_3} = u_{th}$$

5.2.2 Calculation of Thevenin resistance ($\mathbf{R_{th}}$): short circuit analysis

To obtain the value of R_{th} we analyse the behaviour of the equivalent circuit and the original circuit when a short circuit (i.e. an element of resistance 0) is placed between A and B.

In that case, a short circuit current (i_{sc}) flows through the circuit:



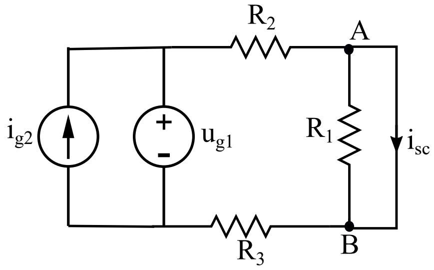
It can be seen that the voltage drop across the resistor R_{th} equals u_{th} (we just have to apply KVL to the circuit). Then, applying Ohm's law:

$$u_{th} = R_{th} \cdot i_{sc}$$

and R_{th} can be calculated as a function of u_{th} and $i_s c$:

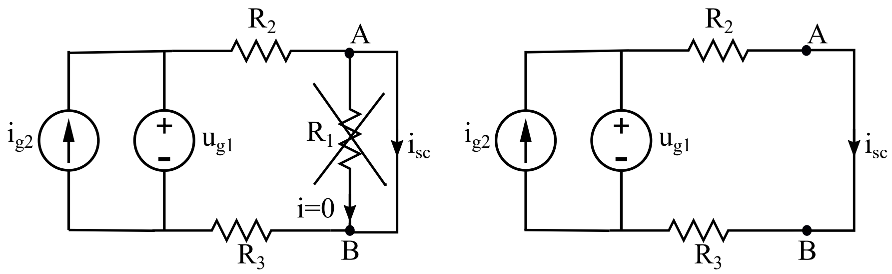
$$R_{th} = \frac{u_{th}}{i_{sc}}$$

To calculate the value of i_{sc} we need to go back to the original circuit. As both circuits are equivalent we may find i_{sc} as the current flowing through a short-circuit placed between A and B:



Our goal is to calculate the current i_{sc} flowing from A to B. We could use any method of analysis to this end.

It is easy to find the value of $i_s c$ if we consider that, as R_1 is now in parallel with a short circuit it can be cancelled [5]



The current i_{sc} is:

$$i_{sc} = \frac{u_{g1}}{R_2 + R_3}$$

and then

$$R_{th} = \frac{u_{th}}{i_{sc}} = \frac{R_1 \cdot u_{g1}}{R_1 + R_2 + R_3} : \frac{u_{g1}}{R_2 + R_3} = \frac{R_1 \cdot (R_2 + R_3)}{R_1 + R_2 + R_3}$$

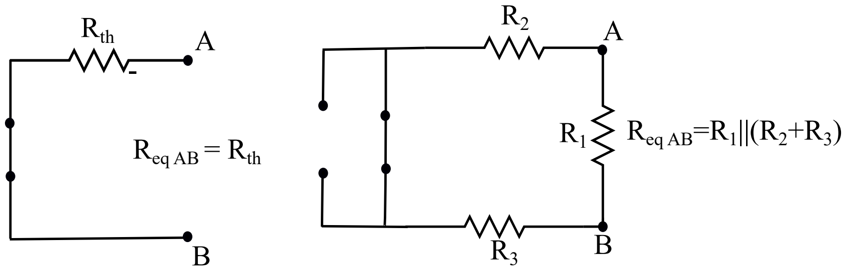
[5]Note that no current flows through a resistor that is connected in parallel with a short circuit because all the current entering to node A would "choose" to flow through the zero-resistance path. Another way to think of it is to consider that now we have a null resistor in parallel with resistor the R_1 and then the obtained equivalent resistance is zero.

5.2.3 Alternative method to calculate $\mathbf{R_{th}}$: passive circuit

An alternative method to calculate R_{th} consists of passivizing the circuit and calculate the equivalent resistance between terminals A and B.

To passivize the circuit we should turn off the sources. As was explained before voltage sources are turned into short circuits and current sources into open circuits.

Then the equivalent circuit and the original circuit would turn into:

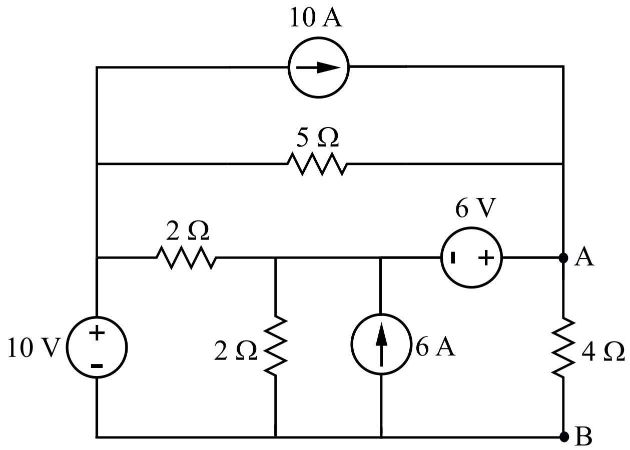


The equivalent resistance between terminals A and B is R_{th}

$$R_{th} = \frac{R_1 \cdot (R_2 + R_3)}{R_1 + R_2 + R_3}$$

5.3 Example 1

Find the Thevenin equivalent of the circuit below between terminals A and B



In this example we are requested to calculate the Thevenin equivalent of the whole circuit (ti.e including all the elements of the circuit in it).

We need to calculate the parameters of the Thevenin equivalent: u_{th} and R_{th}

1. Calculation of Thevenin voltage

   To obtain the value of Thevenin voltage we calculate the voltage drop between terminals A and B in the original circuit (u_{AB}). We might apply mesh or nodal analysis to find that voltage.

In section 3.3 the same circuit was solved using nodal analysis finding that $u_{AB} = 20V$. Then:

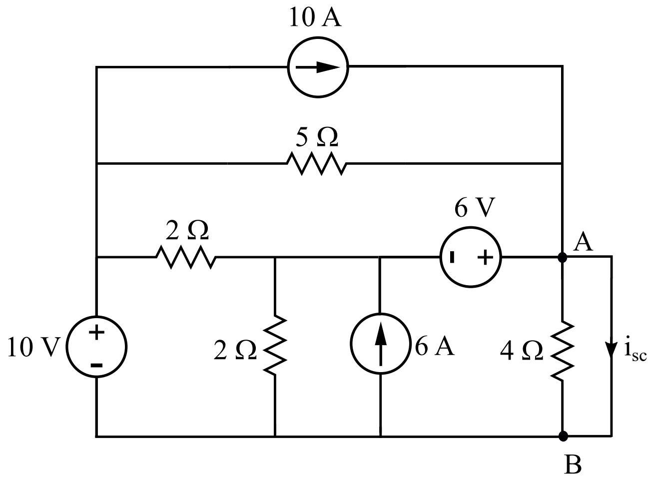
$$u_{th} = u_{AB} = 20V$$

2. Calculation of Thevenin resistance

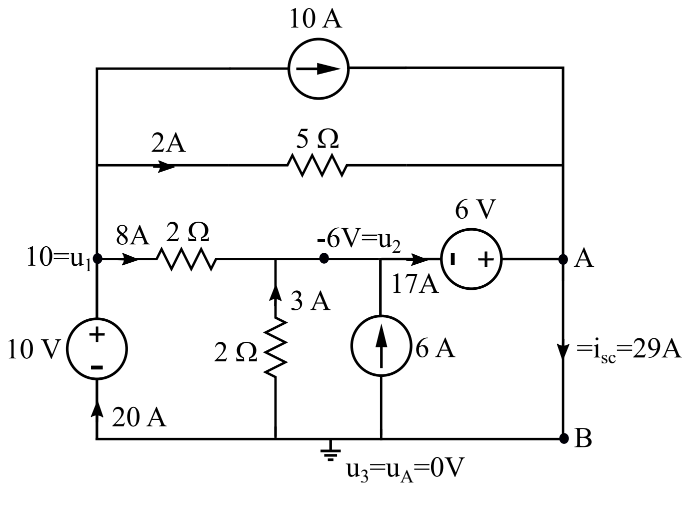
   To calculate R_{th} we have two alternative methods: The calculation of the short-circuit current and the calculation of the equivalent resistance between A B in the passive circuit.

   (a) Method 1: Calculation of i_{sc}

   We place a short-circuit between A and B and calculate the current flowing from A to B (isc).



   We apply nodal analysis to solve the circuit. Since the 4Ω resistor is in parallel with a short-circuit and no current will flow through it and it can be eliminated from the circuit. Taking B as reference node, and considering that now $u_A = u_B$, it is easy to obtain the remaining node voltage and branch currents.



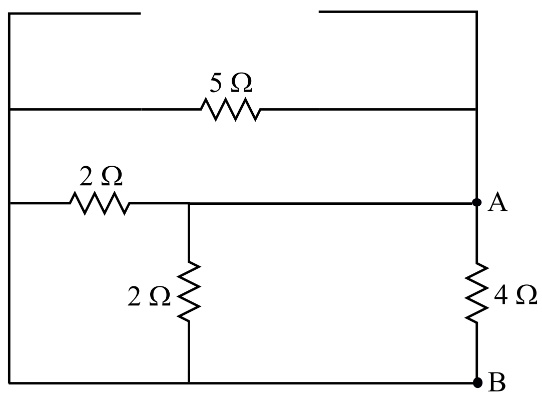
   As can be seen $i_{sc} = 29A$

Finally we calculate R_{th} as

$$R_{th} = \frac{u_{th}}{i_{sc}} = \frac{20}{29} = 0.69\Omega$$

(b) Method 2: Calculation of $R_{eq_{AB}}$

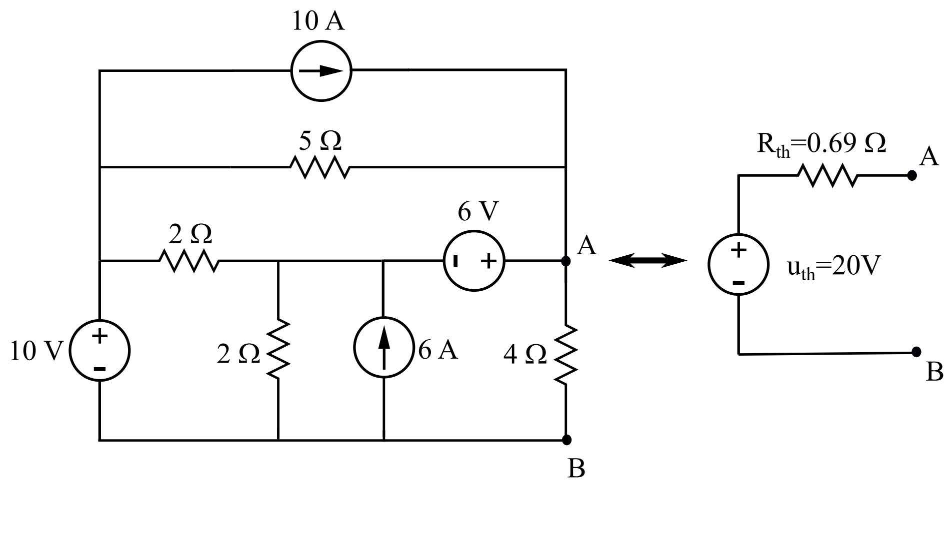
The circuit is passivized turning off the voltage and current sources and finding the following net:



Seen from terminals A B, the four resistors are in parallel. In consequence we calculate Thevenin resistance as:

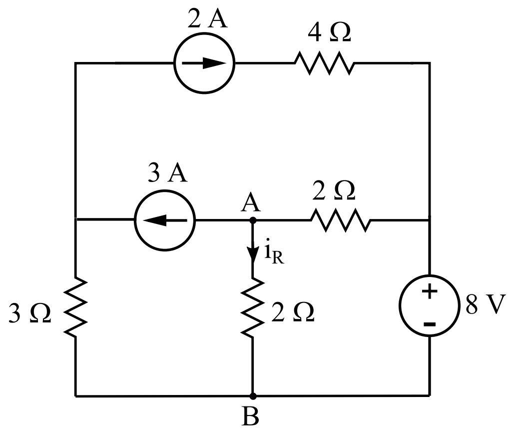
$$R_{th} = R_{eq_{AB}} = 2\Omega||2\Omega||5\Omega||4\Omega = 0.69\Omega$$

The original circuit is equivalent to the simplified Thevenin circuit whose parameters have been obtained in this example:



5.4 Example 2

Use Thevenin's Theorem to calculate the current i_R flowing through the 2 Ω resistor connected between A and B.

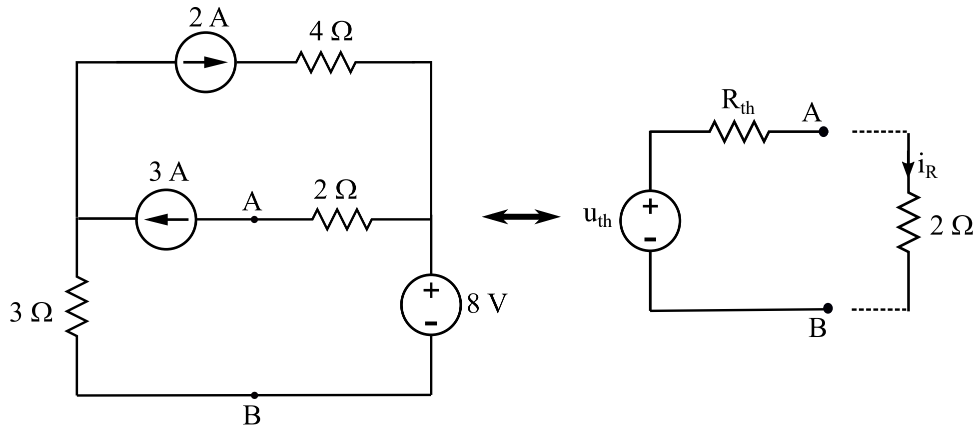


Solution

This second example shows how to apply Thevenin equivalent to the analysis of a particular element of a circuit.

In this case we must not include the resistor that we want to study (the 2Ω one) in the equivalent. Then, we calculate the equivalent of the circuit that results when the resistor is removed.

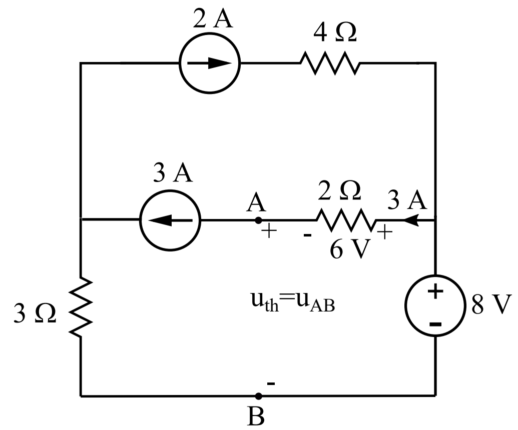
Once obtained the Thevenin's equivalent of the remaining circuit we connect the 2 Ω resistor to the equivalent and calculate the requested current (i_R):



1. Calculation of u_{th}

   Thevenin voltage is the voltage drop between terminals A B of the original circuit after removing the 2 Ω resistor:

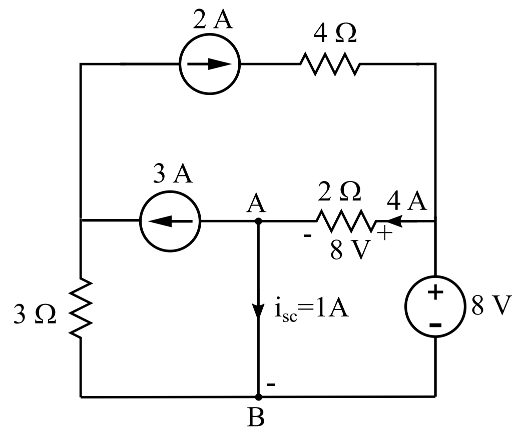
$$u_{th} = u_{AB} = -6 + 8 = 2V$$



2. Calculation of R_{th}

   (a) Method 1: Short circuit current

   We place a short-circuit between A and B and obtain the current that flows through it.

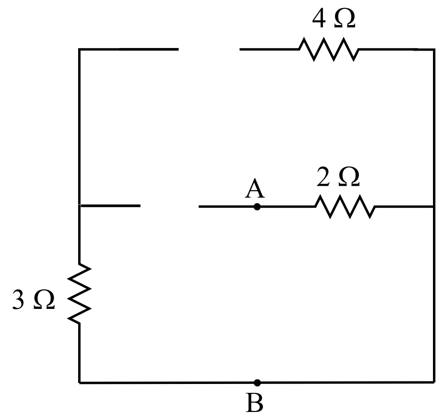


   As can be seen $i_{sc} = 1A$. Then:

$$R_{th} = \frac{u_{th}}{i_{sc}} = \frac{2}{1} = 2\Omega$$

   (b) Method 2: Equivalent resistance

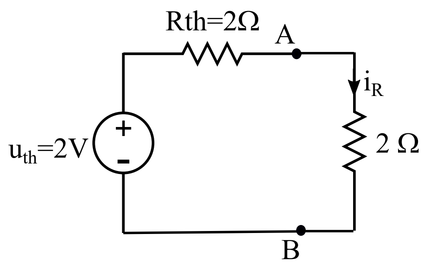
   As an alternative, we can "turn off" the all the sources and calculate the equivalent resistance between terminals A B.



As the resistors of 3 and 4 Ω are in series with open circuits, the only resistor that contributes to the equivalent resistance is the 2 Ω one. Thus:

$$R_{th} = R_{eq_{AB}} = 2\Omega$$

Finally we use the obtained Thevenin equivalent to calculate i_R:

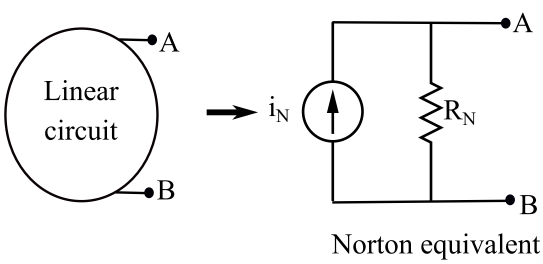


$$i_R = \frac{u_{th}}{R_{th}+2} = \frac{2}{2+2} = 0.5A$$

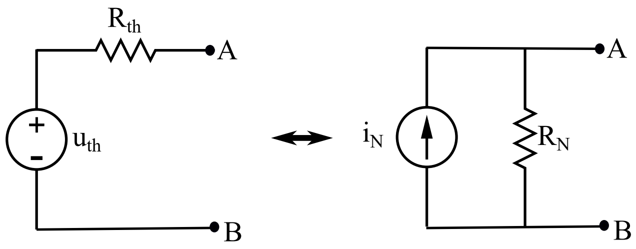
It can be noted that the obtained value of i_R agrees with the calculation carried out in a previous example using mesh analysis (section 2.3).

5.5 Norton's theorem

Norton's theorem states that "any linear circuit can be replaced by an ideal current source in parallel with a resistor".



For the calculation of the Norton equivalent of a circuit the sources transformation rules can be applied.



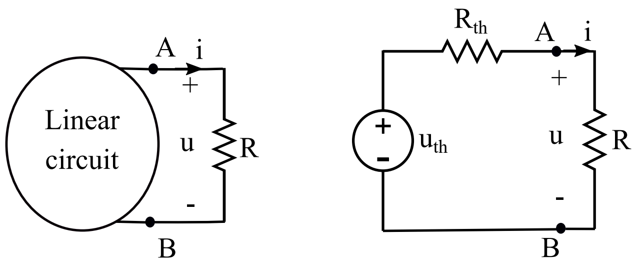
$$R_N = R_{th} \qquad i_N = \frac{u_{th}}{R_{th}}$$

Note that i_N coincides with the short circuit current of the circuit that was defined in section 5.2.2

5.6 Maximum power transference theorem

In some applications we want to determine the value of the resistor that extract the maximum amount of power from a circuit. This is the case of the dessign of antennas or audio systems.

To find the value of that resistor, we could redraw the circuit as its Thevenin equivalent and then calculate the R that absorbs the maximum power from the circuit.



The power supplied to the resistor R is:

$$p = R \cdot i^2 = R \cdot \frac{u_{th}^2}{(R_{th} + R)^2}$$

Maximizing the power we obtain that the resistor that absorbs the maximum amount of power from the circuit coincides with the circuit's Thevenin resistance.

$$\frac{dp}{dR} = 0 \qquad R = R_{th}$$