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# Module 3: Analysis of AC circuits 

Electrical power engineering fundamentals

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## Why AC systems



The line losses depend on the current that flows through the line.

$$
p_{\text {load }}=u \cdot i \quad p_{\text {loss }}=R_{T L} \cdot i^{2}
$$

If the voltage of the system is raised up the line losses cuts

## Transformers

Transformers are used to change the voltage level of electric energy.


Transformers are based in two coupled inductors with different number of turns. The transformation relation is:

$$
r_{t}=\frac{N_{1}}{N_{2}}=\frac{u_{1}}{u_{2}}
$$

## Typical configuration of a power system



## DC vs. $A C$ sources




$$
u(t)=U_{\max } \cdot \cos \left(\omega t+\varphi_{u}\right)
$$

$$
i(t)=I_{\max } \cdot \cos \left(\omega t+\varphi_{i}\right)
$$

## Main parameters of a sinusoidal signal



$$
u(t)=U_{\max } \cdot \cos \left(\omega t+\varphi_{u}\right)
$$

Sinusoidal functions might also be defined in terms of a function sine $(\sin \alpha=\cos (\alpha-\pi / 2))$ but cosine is used in this course

## Main parameters of a sinusoidal signal

- Amplitude (Umax): maximum value reached by the voltage in the whole function range
- Period (T): time needed to complete a cycle (expressed in [s])
- Frequency: number of cycles described in one second.

$$
f=\frac{1}{T} \quad[H z]
$$

- Angular frequency: Frequency of the function in radians per second.

$$
\omega=2 \cdot \pi \cdot f \quad[\mathrm{rad}] \cdot[s]^{-1}
$$

- Phase angle $(\varphi)$ : phase difference between the maximum of the function and the origin (expressed in [rad] and sometimes in degrees).


## Main parameters of a sinusoidal signal

- Mean value: The mean value of a sinusoidal function equals zero

$$
U_{\text {mean }}=\frac{1}{T} \int_{t_{0}}^{t_{0}+T} u(t) d t=\frac{1}{T} \int_{t_{0}}^{t_{0}+T} U_{\max } \cdot \cos \left(\omega t+\varphi_{u}\right) d t=0
$$

- Root mean square value (rms) or effective value:

$$
U_{r m s}=\sqrt{\frac{1}{T} \int_{t_{0}}^{t_{0}+T} u^{2}(t) d t}
$$

## RMS value of a sinusoidal signal

The RMS value of a sinusoidal signal is:

$$
U_{r m s}=\sqrt{\frac{1}{T} \int_{t_{0}}^{t_{0}+T} U_{\max }^{2} \cdot \cos ^{2}\left(\omega t+\varphi_{u}\right) d t}=\frac{U_{\max }}{\sqrt{2}}
$$

The rms value of the sinusoidal signals is a key parameter and it is functions in terms of rms instead of peak values:

$$
\begin{aligned}
U=U_{r m s}=\frac{U_{\max }}{\sqrt{2}} & I=I_{r m s}=\frac{I_{\max }}{\sqrt{2}} \\
u(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right) & i(t)=\sqrt{2} \cdot I \cdot \cos \left(\omega t+\varphi_{i}\right)
\end{aligned}
$$

## Relative phase shift between two signals

Distance between the zero crossings or the peaks of the signals

$$
u(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right) \quad i(t)=\sqrt{2} \cdot I \cdot \cos \left(\omega t+\varphi_{i}\right)
$$


$\Delta \varphi_{u, i}<0$ : voltage lags current and current leads voltage.
$\Delta \varphi_{u, i}=\varphi_{u}-\varphi_{i} \quad \Delta \varphi_{u, i}>0:$ current lags voltage and voltage leads current.

## Relative phase shift between two signals



Current leads voltage
Voltage lags current


Current and voltage are in phase


Voltage leads current
Current lags voltage


Current and voltage are in antiphase

## Challenges of the analysis of AC circuits in the time domain

1. Operating with sinusoidal functions is not easy.


$$
i_{3}=\sqrt{2} \cdot I_{1} \cdot \cos \left(\omega t+\varphi_{i_{1}}\right)+\sqrt{2} \cdot I_{2} \cdot \cos \left(\omega t+\varphi_{i_{2}}\right)=\sqrt{2} \cdot I_{3} \cdot \cos \left(\omega t+\varphi_{i_{3}}\right)
$$

Finding $I_{3}$ and $\varphi_{i_{3}}$ requires complicated math analysis
2. The analysis of AC circuits involves the solution of differential equations or differential systems of equations.

Analysis of a RLC circuit in the time domain


$$
u(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right)
$$

According to KVL:

$$
u_{g}(t)=u_{R}(t)+u_{L}(t)+u_{C}(t)
$$

$$
u_{R}(t)=R \cdot i(t) \quad u_{L}(t)=L \cdot \frac{d i(t)}{d t} \quad u_{C}(t)=\frac{1}{C} \cdot \int i(t) d t
$$

## Analysis of a RLC circuit in the time domain



$$
i(t)=i_{\text {transient }}+i_{\text {steady-state }}
$$

$$
u_{g}(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right) \quad \Rightarrow \quad i_{s s}(t)=\sqrt{2} \cdot l \cdot \cos \left(\omega t+\varphi_{i}\right)
$$

## Some conclusions about the analisis of AC circuits

- The analysis of AC circuits involves the resolution of differential equations or systems of equations. For complex circuits finding and a solution is challenging.
- If the excitation of a circuit is a sinusoidal voltage or current of frequency $\omega$, all the resulting currents and voltages are also sinusoidal functions of the same frequency.
- Our goal is to find the amplitudes and phase shifts of the responses.
- Below we introduce the analysis of AC circuits in the frequency domain which is based in the representation of sinusoidal functions by means of complex numbers.


## Complex numbers: a short review

$$
j=\sqrt{-1}
$$

Complex number:

$$
\begin{gathered}
z=a+b j \\
z=|z| \angle \theta \quad z=|z| \cdot e^{j \theta}
\end{gathered}
$$



Conversion between rectangular and polar form:

$$
|z|=\sqrt{a^{2}+b^{2}} \quad \theta=\arctan \frac{b}{a} \quad a=|z| \cdot \cos \theta \quad b=|z| \cdot \sin \theta
$$

Euler's equation:

$$
e^{ \pm j \theta}=\cos \theta \pm j \sin \theta
$$

## Phasor representation of a sinusoidal function

$$
u(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right)
$$

Applying Euler's equation ( $\left.e^{j \theta}=\cos \theta+j \sin \theta\right)$, $u(t)$ can be expressed as:

$$
u(t)=\sqrt{2} \cdot U \cdot \operatorname{Re}\left(e^{j\left(\omega t+\varphi_{u}\right)}\right)=\sqrt{2} \cdot \operatorname{Re}\left(U \cdot e^{j \varphi_{u}} \cdot e^{j \omega t}\right)
$$

The important information of $u(t)$ is $U$ and $\varphi_{u}$, since $\omega$ is the same for all the currents and voltages of the circuit.

The phasor representation of a sinusoidal signal is a complex number that contains information on the rms value and the phase angle of the sinusoidal function.

$$
\underline{\mathbf{U}}=U \cdot e^{j \varphi_{u}}=U \angle \varphi_{u}
$$

## Phasor representation of a sinusoidal function

Phasors represent sinusoidal functions in the frequency domain

Time domain


$u(t)=\sqrt{2} \cdot \operatorname{Re}\left(U \cdot e^{j \varphi_{u}} \cdot e^{j \omega t}\right)$
$\underline{\mathbf{U}}=U \cdot e^{j \varphi_{u}}=U \angle \varphi_{u}$

$$
u(t)=\sqrt{2} \cdot \operatorname{Re}\left(\underline{\mathbf{U}} \cdot e^{j \omega t}\right)
$$

## Kirchhoff's laws in phasor form

It can be proved that KCL and KVL are also verified for the phasors associated to voltages and currents.

$$
i_{k}(t)=\sqrt{2} \cdot I_{k} \cdot \cos \left(\omega t+\varphi_{i, k}\right) \quad \underline{\mathbf{I}_{k}}=I_{k} \angle \varphi_{i, k}
$$

Kirchhoff's current law:

$$
\sum_{k=1}^{n} i_{k}(t)=0
$$

$$
u_{k}(t)=\sqrt{2} \cdot U_{k} \cdot \cos \left(\omega t+\varphi_{u, k}\right)
$$

$$
\underline{\mathbf{U}_{k}}=U_{k} \angle \varphi_{u, k}=0
$$

Kirchhoff's voltage law :

$$
\sum_{k=1}^{n} u_{k}(t) \quad \sum_{k=1}^{n} \underline{\mathbf{U}_{k}}=0
$$

## Example

Calculate $i_{3}(t)$ given that:
$i_{1}(t)=\sqrt{2} \cdot 10 \cdot \cos \left(25 t+45^{\circ}\right) A \quad i_{2}(t)=\sqrt{2} \cdot 20 \cdot \cos \left(25 t+90^{\circ}\right) A$


## Solution

$$
\begin{gathered}
\underline{\mathbf{I}}_{1}=10 \angle 45^{\circ} A=7.07+7.07 j A \quad \underline{\mathbf{I}}_{2}=20 \angle 90^{\circ} A=20 j A \\
\underline{\mathbf{I}}_{3}=\underline{\mathbf{I}}_{1}+\underline{\mathbf{I}}_{2}=10 \angle 45^{\circ}+20 \angle 90^{\circ}=7.07+27.07 j=28 \angle 75.36^{\circ} A \\
i_{3}(t)=\sqrt{2} \cdot 28 \cdot \cos \left(25 t+75.36^{\circ}\right) A
\end{gathered}
$$

## Impedance of a passive element

For capacitors and inductors, the relationships between voltages and currents are given by differential equation.

$$
u_{L}(t)=L \cdot \frac{d i_{L}(t)}{d t} \quad i_{C}(t)=C \cdot \frac{d u_{C}(t)}{d t} \quad u_{R}(t)=R \cdot i_{R}(t)
$$

To avoid the need of solving differential equations in the analysis of $A C$ circuits we define the impedance of a passive element:

$$
Z=\frac{\underline{\mathbf{U}}}{\underline{\mathbf{I}}} \quad[V] \cdot[A]^{-1}=[\Omega]
$$

Admitance: Inverse of the impedance:

$$
Y=\frac{1}{Z} \quad[\Omega]^{-1}=[S]
$$

## Ohm's law in the frequency domain

We represent the three types of passive elements as impedances.


Ohm's law in the frequency domain: $\underline{\mathbf{U}}=Z \cdot \underline{I}$

## Impedance of a resistor



$$
u_{R}(t)=R \cdot i_{R}(t)
$$

Time domain:

$$
\begin{gathered}
i_{R}(t)=\sqrt{2} \cdot I_{R} \cdot \cos \left(\omega t+\varphi_{i}\right) \\
u_{R}(t)=\sqrt{2} \cdot R \cdot I_{R} \cdot \cos \left(\omega t+\varphi_{i}\right)
\end{gathered}
$$

Frequency domain:
$\underline{\mathbf{I}}_{R}=I_{R} \angle \varphi_{i} \quad \underline{\mathbf{U}}_{R}=R \cdot I_{R} \angle \varphi_{i} \quad Z_{R}=\frac{\underline{\mathbf{U}}_{R}}{\underline{\mathbf{I}}_{R}}=\frac{R \cdot I_{R} \angle \varphi_{i}}{I_{R} \angle \varphi_{i}}=R$

Impedance of a resistor: $\quad \mathrm{Z}_{R}=R$

## Resistors in the time and frequency domain

Time domain



Frequency domain



Voltages and currents in resistors are in phase:

$$
\varphi_{i}=\varphi_{u} \quad \Delta \varphi_{u, i}=0
$$

## Impedance of an inductor

$$
\left.\vec{i}_{+}^{+} \mathrm{u}_{\mathrm{L}}(\mathrm{t}) \quad \mathrm{L}\right)--
$$

$$
u_{L}(t)=L \cdot \frac{d i_{L}(t)}{d t}
$$

Time domain:

$$
\begin{gathered}
i_{L}(t)=\sqrt{2} \cdot I_{L} \cdot \cos \left(\omega t+\varphi_{i}\right) \\
u_{L}(t)=-\sqrt{2} \cdot \omega \cdot L \cdot I_{L} \cdot \sin \left(\omega t+\varphi_{i}\right)=-\sqrt{2} \cdot \omega \cdot L \cdot I_{L} \cdot \cos \left(\omega t+\varphi_{i}-\frac{\pi}{2}\right)
\end{gathered}
$$

Frequency domain:

$$
\underline{\mathbf{I}}_{L}=I_{L} \angle \varphi_{i} \quad \underline{\mathbf{U}}_{L}=\omega \cdot L \cdot I_{L} \angle \varphi_{i}+\frac{\pi}{2} \quad Z_{L}=\frac{\omega \cdot L \cdot I_{L} \angle \varphi_{i}+\frac{\pi}{2}}{I_{L} \angle \varphi_{i}}=j \omega L
$$

## Inductors in the time and frequency domain

Time domain
Frequency domain




In inductors voltage lead currents by $90^{\circ}$.

$$
\varphi_{u}=\varphi_{i}+\frac{\pi}{2} \quad \Delta \varphi_{u, i}=\frac{\pi}{2}=90^{\circ}
$$

## Impedance of a capacitor



$$
i(t)=C \cdot \frac{d u(t)}{d t}
$$

Time domain:

$$
\begin{gathered}
u_{C}(t)=\sqrt{2} \cdot U_{C} \cdot \cos \left(\omega t+\varphi_{u}\right) \\
i(t)=-\sqrt{2} \cdot \omega \cdot C \cdot U_{C} \cdot \sin \left(\omega t+\varphi_{u}\right)=\sqrt{2} \cdot \omega \cdot C \cdot U_{C} \cdot \cos \left(\omega t+\varphi_{u}-\frac{\pi}{2}+\pi\right)
\end{gathered}
$$

Frequency domain:

$$
\underline{\mathbf{u}}_{C}=U_{C} \angle \varphi_{u} \quad \underline{\mathbf{l}}_{C}=j \omega C U_{C} \angle \varphi_{u} \quad Z_{C}=\frac{U_{C} \angle \varphi_{u}}{j \omega C U_{C} \angle \varphi_{u}}=\frac{-j}{\omega C}
$$

Impedance of a capacitor: $\quad Z_{C}=-j / \omega C$

## Capacitors in the time and frequency domain

Time domain




In capacitor current lead voltage by $90^{\circ}$.

$$
\varphi_{i}=\varphi_{u}+\frac{\pi}{2} \quad \Delta \varphi_{u, i}=-\frac{\pi}{2}=-90^{\circ}
$$

## Association of impedances

- Association of passive elements of different nature is not possible in the time domain:

$$
u_{R}(t)=R \cdot i_{R}(t) \quad u_{L}(t)=L \cdot \frac{d i_{L}(t)}{d t} \quad i_{C}(t)=C \cdot \frac{d u_{C}(t)}{d t}
$$

- In the frequency domain the relation between $\underline{\mathbf{U}}$ and $\underline{\mathbf{I}}$ is the same for $R, L$ and $C$ :

$$
\underline{\mathbf{U}}=Z \cdot \underline{\mathbf{l}}
$$

- The representation of $\mathrm{R}, \mathrm{L}$ and C with impedances makes possible the association of passive elements of different nature.


## Association of impedances in series

Two or more impedances are series connected if the same phasor current flows through them:


The set of n impedances can be redrawn as an equivalent impedance $Z_{\text {eq }}$

$$
Z_{e q}=Z_{1}+Z_{2}+\ldots .+Z_{n}=\sum_{k=1}^{n} Z_{k}
$$

## Voltage divider equation



The voltage divider equation is also valid in the frequency domain. The phasor voltage across impedance $k$ is:

$$
\underline{\mathbf{u}}_{k}=\frac{Z_{k}}{Z_{e q}} \cdot \underline{\mathbf{u}}
$$

## Association of impedances in parallel

Two or more impedances are parallel connected if they have the same phasor voltage across them:


The set of n impedances can be redrawn as an equivalent impedance $Z_{\text {eq }}$

$$
\frac{1}{Z_{e q}}=\frac{1}{Z_{1}}+\frac{1}{Z_{2}}+\ldots+\frac{1}{Z_{n}}=\sum_{k=1}^{n} \frac{1}{Z_{k}} \quad Y_{e q}=\sum_{k=1}^{n} Y_{k}
$$

## Current divider equation



The current divider equation is also valid in the frequency domain. The phasor curretn through across impedance $k$ is:

$$
\underline{\mathbf{I}}_{k}=\frac{Y_{k}}{Y_{e q}} \cdot \underline{\mathbf{l}}
$$

* For the case of two impedances in parallel:

$$
\underline{\mathbf{l}}_{1}=\frac{Y_{1}}{Y_{1}+Y_{2}} \cdot \underline{\mathbf{l}}=\frac{Z_{2}}{Z_{1}+Z_{2}} \cdot \underline{\mathbf{l}} \quad \underline{\mathbf{l}}_{2}=\frac{Y_{2}}{Y_{1}+Y_{2}} \cdot \underline{\mathbf{I}}=\frac{Z_{1}}{Z_{1}+Z_{2}} \cdot \underline{\mathbf{l}}
$$

## Complex impedance

- The impedances that represent the three passive elements are:

$$
Z_{R}=R \in \mathbb{R} \quad Z_{L}=j \omega \cdot L \in \mathbb{C} \quad Z_{C}=\frac{-j}{\omega \cdot C} \in \mathbb{C}
$$

- If we add two or more elements of different nature may find impedances with real and imaginary part:

$$
Z=R+j X
$$

- The real part of the impedance always comes from resistive elements, and it is called the Resistance (R).
- The imaginary part always comes from inductors and capacitors and is called the Reactance ( X ).


## Components of a complex impedance

Impedance triangle:

$$
\begin{aligned}
Z & =R+j X \\
\varphi & =\arctan \frac{X}{R}
\end{aligned}
$$

The power factor of an impedance is the cosine of $\varphi$

Resistance Reactance Impedance
Resistor
Inductor
0
$\omega \cdot L$
$j \omega \cdot L$
Capacitor
0
$-1 / \omega \cdot C \quad-j / \omega \cdot C$

## Mesh current method

1. Assign a phasor mesh current to each mesh of the circuit
2. Apply 2 KL in phasor form to every mesh of the circuit applying a consistent sign criteria and find a system of equations with the mesh currents as unknowns.

Mesh equations in matrix form are:

$$
[Z] \cdot\left[\underline{I}_{\text {mesh }}\right]=\left[\underline{\mathbf{U}}_{g}\right]
$$

[ $Z$ ] is the impedance matrix:
$\mathrm{Zii}=$ Sum of the impedances in mesh i
$\mathrm{Zij}=-$ Sum of the impedances shared by mesh i and j
3. Solve the equations to find the currents.

## Node voltage method

1. Assign a phasor node voltage to each node of the circuit and label one of them as reference node.
2. Apply KCL in phasor form to each node of the circuit with a consistent sign criteria

Nodal equations in matrix form are:

$$
[Y] \cdot\left[\underline{\mathbf{U}}_{\text {node }}\right]=\left[\underline{\mathbf{I}}_{g}\right]
$$

[ $Y$ ] is the Admitance matrix whose terms are:
Yii=Sum of the admitances connected to node i
$Y i j=-$ Sum of the admitances shared by nodes $i$ and $j$
3. Solve the equations to find the node voltages.

## Thevenin equivalent

Thenvenin's theorem is valid for AC circuits in the frequency domain.

The Thevenin equivalent of a circuit in the frequency domain consists of a voltage source of value $\underline{\mathbf{U}}_{t h}$ and a series impedance $Z_{t h}$.


The methods that can be followed to obtain the parameters of the equivalent are analogous to the ones studied for DC circuits.

## Superposition principle for the analysis of circuits with

 sources of different frequency

Which frequency should we consider for the calculation of the impedances?

Superposition principle: "The response of a linear circuit subjected to several excitation sources acting simultaneously equals the sum of the responses of the circuits when the sources act separately".

## Application of the method: stage 1

1. Turn off all the sources of frequency $\omega_{2}$. Replace the voltage sources by short circuits and the current sources by open circuits.
2. Transform the circuit into the frequency domain considering the frequency $\omega_{1}$ to calculate the impedances of the passive elements.
3. Calculate the response of the circuit in the frequency domain (i.e. the phasors current and voltage at the different parts of the circuit)
4. Find the instantaneous currents and voltages in the time domain which constitute the response of the circuit to the sources of frequency $\omega_{1}$.

## Application of the method: stage 2

1. Turn off all the sources of frequency $\omega_{1}$.
2. Recalculate the impedances of the passive elements for the frequency $\omega_{2}$ and obtain the currents and voltages of the elements in the frequency domain
3. Obtain the currents and voltages in the time domain.
4. Obtain the response of the system when all the sources act simultaneously as the sum of the separate responses in the time domain. The currents and voltages will be a sum of two sinusoidal functions of frequencies $\omega_{1}$ and $\omega_{2}$.

## Example



## Instantaneous power

Imagine that we want to calculate the power absorbed by the RLC net shown in the diagram:


$$
u(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+\varphi_{u}\right) \quad i(t)=\sqrt{2} \cdot I \cdot \cos \left(\omega t+\varphi_{i}\right)
$$

The phase angle between $\mathrm{u}(\mathrm{t})$ and $\mathrm{i}(\mathrm{t})$ is: $\varphi=\varphi_{u}-\varphi_{i}$
For simplicity we take the current as phase origin:

$$
u(t)=\sqrt{2} \cdot U \cdot \cos (\omega t+\varphi) \quad i(t)=\sqrt{2} \cdot l \cdot \cos (\omega t)
$$

## Instantaneous power of an AC circuit

The instantaneous electric power is the product between $u(t)$ and i(t):

$$
p(t)=u(t) \cdot i(t)=2 \cdot U \cdot l \cdot \cos (\omega t+\varphi) \cdot \cos (\omega t)
$$

using: $\cos \alpha \cdot \cos \beta=\frac{1}{2} \cdot(\cos (\alpha+\beta)+\cos (\alpha-\beta))$

$$
p(t)=U \cdot I \cdot \cos \varphi+U \cdot I \cdot \cos (2 \omega t+\varphi)
$$

using: $\cos (\alpha+\beta)=\cos \alpha \cdot \cos \beta-\sin \alpha \cdot \sin \beta$

$$
p(t)=U \cdot I \cdot \cos \varphi+U \cdot I \cdot(\cos 2 \omega t \cdot \cos \varphi-\sin 2 \omega t \cdot \sin \varphi)
$$

Rearranging:

$$
\mathrm{p}(\mathrm{t})=\mathrm{U} \cdot / \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t
$$

## Instantaneous power of an AC circuit

$$
p(t)=U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t
$$

$$
\mathrm{p}(\mathrm{t}) \text { is sometimes }+ \text { and sometimes - }
$$



## Understanding the instantaneous power

$$
p(t)=U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t
$$

We analyse the two terms of the power separately:

$$
U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)
$$

$$
-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t
$$




Average value $U \cdot I \cdot \cos \varphi$
Amplitude U $\cdot / \cdot \sin \varphi$

## Understanding the instantaneous power

- The instantaneous power fluctuates with frequency $2 \omega$ being sometimes positive and sometimes negative
- $\mathrm{p}(\mathrm{t})$ is made of two terms:

1. One is always positive and of average vale $U \cdot I \cdot \cos \varphi$.

This term is the power absorbed in resistors

$$
P=U \cdot l \cdot \cos \varphi \quad \text { Active power }
$$

2. The other one is sometimes positive and sometimes negative with average value and amplitude $U \cdot I \cdot \sin \varphi$

This term is the power exchanged with the source by inductors and capacitors

$$
Q=U \cdot I \cdot \sin \varphi \quad \text { Reactive power }
$$

## Instantaneous power of a resistor

$$
\begin{gathered}
p(t)=U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t \\
\varphi_{R}=\varphi_{u}-\varphi_{i}=0 \quad p_{R}(t)=U \cdot I \cdot(1+\cos 2 \omega t)
\end{gathered}
$$



## Instantaneous power of a resistor

The instantaneous power of a resistor is:

$$
p_{R}(t)=U \cdot I \cdot(1+\cos 2 \omega t)
$$

Remarks:

- The power is always positive. This is consistent with what we studied before, because resistors always dissipate energy.
- The power fluctuates with frequency $2 \omega$
- The average value of the power is $U \cdot I$


## Instantaneous power of an inductor

$$
\begin{gathered}
p(t)=U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t \\
\varphi_{L}=\varphi_{u}-\varphi_{i}=90^{\circ} \quad p_{L}(t)=-U \cdot I \cdot \sin 2 \omega t
\end{gathered}
$$



## Instantaneous power of an inductor

The instantaneous power of an inductor is:

$$
p_{L}(t)=-U \cdot I \cdot \sin 2 \omega t
$$

Remarks:

- The power of an inductor fluctuates with frequency $2 \omega$ being sometimes positive and sometimes negative.
- The average value of the instantaneous power is zero.
- These observations are consistent with what we studied about power of inductors: Inductors do not dissipate power, but they store energy in a magnetic field.
- In the first part of the cycle the energy is stored, and in the second part the energy is released and returned to the source.


## Instantaneous power of a capacitor

$$
\begin{gathered}
p(t)=U \cdot I \cdot \cos \varphi \cdot(1+\cos 2 \omega t)-U \cdot I \cdot \sin \varphi \cdot \sin 2 \omega t \\
\varphi_{C}=\varphi_{u}-\varphi_{i}=-90^{\circ} \quad p_{C}(t)=U \cdot I \cdot \sin 2 \omega t
\end{gathered}
$$



## Instantaneous power of a capacitor

The instantaneous power of a capacitor is:

$$
p_{C}(t)=U \cdot l \cdot \sin 2 \omega t
$$

Remarks:

- Power fluctuates with frequency $2 \omega$ being sometimes positive and sometimes negative, what means that the capacitor absorbs and delivers power alternatively.
- The average value of the instantaneous power is zero.
- The observations are consistent with the fact that capacitors do not dissipate power, but store energy in an electric field instead.
- In the first part of the cycle the energy is stored and then the energy is released and returned to the source.


## Power absorbed vs. power exchanged

Two different behaviours in relation to energy in AC circuits:

1. Resistors absorb power from the sources and transform it into heat.
2. Inductors and capacitors absorb power from the sources, store it and return it later.

The absorbed power and the fluctuating power are both important and must be taken into account in the analysis of AC circuits.

Nevertheless they have a different impact on circuits because the first supposes a real consumption of energy the second is a continuous flow of energy between the source and the inductive and capacitive loads.

## Active and reactive power

Two types of power in AC circuits:

- Active power (P): Power absorbed in resistors.

$$
P=U \cdot I \cdot \cos \varphi
$$

Active power is measured in Watts [W]

- Reactive power (Q): is the power that fluctuates between the inductors and capacitors and the sources.

$$
Q=U \cdot I \cdot \sin \varphi
$$

Reactive power is measured in Volt-Ampere reactive [var]

- Complex power $(\mathcal{S})$ :

$$
\mathcal{S}=P+Q j \quad[V A]
$$

## Power triangle and other power-related parameters



Apparent power (S): Is the modulus of the complex power:

$$
\begin{equation*}
S=\sqrt{P^{2}+Q^{2}}=U \cdot I \tag{VA}
\end{equation*}
$$

Power factor: cosine of the phase shift between voltage and current.

$$
\text { p.f. }=\cos \varphi=\cos \left(\varphi_{u}-\varphi_{i}\right)
$$

## Practical importance of the power factor



- Inductors and capacitors do not absorb power, but they exchange it with the source
- However the flow of energy overloads the lines and produces losses
- Electric companies charge not for the active power but for the apparent power consumed.
- Ideally the power factor must be $1(\mathrm{Q}=0)$


## Power of a complex impedance



Active and reactive power

$$
P=U \cdot I \cdot \cos \varphi=R \cdot I^{2} \quad Q=U \cdot I \cdot \sin \varphi=X \cdot I^{2}
$$

Complex power

$$
\mathcal{S}=P+Q j=Z \cdot I^{2}
$$

## Complex power of AC souces

Sign criteria for power: A source delivers power when current flows from the terminal at lower voltage towards the terminal at higher voltage. Power delivered by sources is taken as positive.


$$
\mathcal{S}_{g}=\underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^{*}=P_{g}+Q_{g} j
$$

The real part of the complex power is the active power delivered by the source and the imaginary part is the reactive power delivered by the source. In some cases $P_{g}$ or $Q_{g}$ might be negative, what means that the source absorbs active or reactive power.

## Boucherot theorem: power balance in AC circuits

The total amount of active and reactive power absorbed in an electric circuit equals the sum of the active and reactive power absorbed by its passive elements.

$$
P_{T}=\sum_{k} P_{k} \quad Q_{T}=\sum_{k} Q_{k}
$$

This implies that the sum of the complex power supplied by the sources equals the sum of the complex power absorbed by the impedances.

## Example



The power supplied by the source equals the sum of the complex power absorbed by the impedances $Z_{1}$ and $Z_{2}$.

$$
\mathcal{S}_{g}=\underline{\mathbf{U}} \cdot \underline{\mathbf{I}}^{*}=\underline{\mathbf{U}} \cdot\left(\underline{\mathbf{I}}_{1}+\underline{\mathbf{I}}_{2}\right)^{*}=S_{Z_{1}}+S_{Z_{2}}=\left(P_{1}+P_{2}\right)+j\left(Q_{1}+Q_{2}\right)
$$

