# Module 4: Analysis of three-phase systems 

Electrical power engineering fundamentals

Belén García<br>Electrical Engineering Department Universidad Carlos III de Madrid

## Configuration of three-phase systems

- Most power systems are three-phase.
- Each part of the system is called phase



## Three phase generators

Three phase generators generate three sinusoidal voltages of the same amplitude and phase shift $120^{\circ}$


$$
\begin{gathered}
u_{a}(t)=\sqrt{2} \cdot U \cdot \cos (\omega t) \\
u_{b}(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t-120^{\circ}\right) \\
u_{c}(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+120^{\circ}\right)
\end{gathered}
$$

## Generators in wye and delta

We represent three-phase generators as three AC sources connected in wye or delta


## Positive and negative phase sequence



Positive sequence

$$
\begin{gathered}
\underline{\mathbf{U}_{a}}=U \angle 0^{\circ} \\
\underline{\mathbf{U}_{b}}=U \angle-120^{\circ} \\
\underline{\mathbf{U}_{c}}=U \angle 120^{\circ}
\end{gathered}
$$



Negative sequence

$$
\begin{gathered}
\underline{\mathbf{U}_{a}}=U \angle 0^{\circ} \\
\underline{\mathbf{U}_{b}}=U \angle 120^{\circ} \\
\underline{\mathbf{U}_{c}}=U \angle-120^{\circ}
\end{gathered}
$$

## Three-phase loads

- Three-phase loads are represented in the frequency domain, as a set of three impedances connected between them.
- We limit our analysis to balanced systems in which the impedance connected to each phase has the same value
- Loads can be connected in wye or delta



## Balanced wye-wye system systems

In wye wye systems the neutral points of the generators and the loads might be connected by means of a neutral wire.


The current of each phase flows from the generator towards the load and returns through the neutral wire

## Currents in a balanced wye-wye system

$$
\underline{\mathbf{I}_{a}}=\frac{\mathbf{U}_{a}}{Z_{Y}}=\frac{U \angle 0}{Z_{Y}} \quad \underline{\mathbf{I}_{b}}=\frac{\mathbf{U}_{b}}{Z_{Y}}=\frac{U \angle-120^{\circ}}{Z_{Y}} \quad \underline{\mathbf{I}_{c}}=\frac{\mathbf{U}_{c}}{Z_{Y}}=\frac{U \angle 120^{\circ}}{Z_{Y}}
$$

The currents also form a three phase system. If $Z_{Y}=\left|Z_{Y}\right| \angle \theta$


Inductive load


Capacitive load

## Current through the neutral wire

As the current flowing through the neutral is zero the wire is often suppressed.

$$
\underline{\mathbf{I}_{N}}=\underline{\mathbf{I}_{a}}+\underline{\mathbf{I}_{b}}+\underline{\mathbf{I}_{c}}=\frac{U}{Z_{Y}} \cdot\left(1 \angle 0+1 \angle-120^{\circ}+1 \angle 120^{\circ}\right)=0
$$



## Savings of three-phase system

- Lower material investment
- Lower line losses for the same power transfer


## Line and phase voltages and currents

- Phase-voltage: Voltage drop across a single phase of the generator or the load. The phase-voltage of the generator is the voltage drop across the terminals of one of the ideal voltage sources; the phase voltage of the load is the voltage drop across one of the impedances that constitutes the load.
- Line-votage: Voltage drop between any couple of lines. We could obtain the line voltage at the generator side of the system, or the line voltage at the load side.
- Phase current: Current in a single phase, i.e current flowing through one of the ideal sources or through one of the impedances.
- Line current: Current in a single line


## Line and phase voltages and currents



Relation between the line and phase magnitudes in a wye-wye system

Relation between the line and phase voltages:

$$
\underline{\mathbf{I}}_{L}=\underline{\mathbf{I}}_{P h}
$$

Relation between the line and phase voltages:

$$
\underline{\mathbf{U}}_{P h a}=\underline{\mathbf{U}}_{a} \quad \underline{\mathbf{U}}_{P h b}=\underline{\mathbf{U}}_{b} \quad \underline{\mathbf{U}}_{P h c}=\underline{\mathbf{U}}_{c}
$$

$$
\begin{gathered}
\underline{\mathbf{U}}_{L a}=\underline{\mathbf{U}}_{a}-\underline{\mathbf{U}}_{b}=U \angle 0-U \angle-120^{\circ}=\sqrt{3} \cdot U \angle 30^{\circ}=\sqrt{3} \cdot \underline{\mathbf{U}}_{P h a} \angle 30^{\circ} \\
\underline{\mathbf{U}}_{L b}=\underline{\mathbf{U}}_{B C}=\underline{\mathbf{U}}_{b}-\underline{\mathbf{U}}_{c}=\sqrt{3} \cdot \underline{\mathbf{U}}_{P h b} \angle 30^{\circ} \\
\underline{\mathbf{U}}_{L c}=\underline{\mathbf{U}}_{C A}=\underline{\mathbf{U}}_{c}-\underline{\mathbf{U}}_{a}=\sqrt{3} \cdot \underline{\mathbf{U}}_{P h c} \angle 30^{\circ}
\end{gathered}
$$

Relation between the line and phase magnitudes in a wye-wye system


The line voltages are $\sqrt{3}$ times larger that the phase voltages and lead the phase voltages by $30^{\circ}$

Relation between the line and phase magnitudes in a delta-delta system


Relation between the line and phase magnitudes in a delta-delta system

## Phase and line voltages:

$$
\underline{\mathbf{U}}_{\text {LINE }}=\underline{\mathbf{U}}_{\text {PHASE }}
$$

Phase currents:

$$
\begin{gathered}
\underline{\mathbf{I}}_{P h a}=\underline{\mathbf{I}}_{B^{\prime} C^{\prime}}=\frac{\underline{\mathbf{U}}_{a}}{Z_{\Delta}}=\frac{U \angle 0}{Z_{\Delta}} \\
\underline{\mathbf{I}}_{P h b}=\underline{\mathbf{I}}_{A^{\prime} B^{\prime}}=\frac{\mathbf{U}_{b}}{Z_{\Delta}}=\frac{U \angle-120^{\circ}}{Z_{\Delta}} \\
\underline{\mathbf{I}}_{P h c}=\underline{\mathbf{I}}_{C^{\prime} A^{\prime}}=\frac{\mathbf{U}_{c}}{Z_{\Delta}}=\frac{U \angle 120^{\circ}}{Z_{\Delta}}
\end{gathered}
$$

Relation between the line and phase magnitudes in a delta-delta system

Line currents:

$$
\begin{gathered}
\underline{\mathbf{I}}_{L a}=\underline{\mathbf{I}}_{P h a}-\underline{\mathbf{I}}_{P h b}=\frac{U \angle 0}{Z_{\Delta}}-\frac{U \angle-120^{\circ}}{Z_{\Delta}}=\sqrt{3} \cdot \frac{U}{Z_{\Delta}} \angle-30^{\circ}=\sqrt{3} \cdot \underline{\mathbf{I}}_{P h a} \angle-30^{\circ} \\
\underline{\mathbf{I}}_{L b}=\sqrt{3} \cdot \underline{\mathbf{I}}_{P h b} \angle-30^{\circ} \\
\underline{\mathbf{I}}_{L c}=\sqrt{3} \cdot \underline{\mathbf{I}}_{P h c} \angle-30^{\circ}
\end{gathered}
$$

The line currents are $\sqrt{3}$ times larger that the phase currents and lag the phase currents by $30^{\circ}$

## Summary



Delta configuration

$$
\begin{gathered}
U_{\text {line }}=U_{\text {phase }} \\
I_{\text {line }}=\sqrt{3} \cdot I_{\text {phase }} \\
\underline{I}_{L A}=\sqrt{3} \cdot \underline{I}_{p h A} \angle-30
\end{gathered}
$$

## Wye configuration

$$
\begin{gathered}
U_{\text {line }}=\sqrt{3} \cdot U_{\text {phase }} \\
l_{\text {line }}=I_{\text {phase }} \\
\underline{\mathbf{U}}_{L A}=\sqrt{3} \cdot \underline{\mathbf{U}}_{p h A} \angle 30
\end{gathered}
$$



Analysis of three-phase systems: one-phase equivalent


$$
\begin{aligned}
& -\underline{\mathbf{U}}_{a}+\underline{\mathbf{I}}_{a} \cdot\left(Z_{T L}+Z_{Y}\right)+\left(\underline{\mathbf{I}}_{a}+\underline{\mathbf{I}}_{b}+\underline{\mathbf{I}}_{c}\right) \cdot Z_{N}=0 \\
& -\underline{\mathbf{U}}_{b}+\underline{\mathbf{I}}_{b} \cdot\left(Z_{T L}+Z_{Y}\right)+\left(\underline{\mathbf{l}}_{a}+\underline{\mathbf{I}}_{b}+\underline{\mathbf{I}}_{c}\right) \cdot Z_{N}=0 \\
& -\underline{\mathbf{U}_{c}}+\underline{\mathbf{I}_{c}} \cdot\left(Z_{T L}+Z_{Y}\right)+\left(\underline{\mathbf{I}_{a}}+\underline{\mathbf{I}_{b}}+\underline{\mathbf{I}}_{c}\right) \cdot Z_{N}=0
\end{aligned}
$$

## Analysis of three-phase circuits



As there is no current flow through the neutral wire, the three phases can be analysed as independent circuits

$$
\begin{aligned}
& -\underline{\mathbf{U}_{a}}+\underline{\mathbf{I}_{a}} \cdot\left(Z_{T L}+Z_{Y}\right)=0 \\
& -\underline{\mathbf{U}_{b}}+\underline{\mathbf{I}_{b}} \cdot\left(Z_{T L}+Z_{Y}\right)=0 \\
& -\underline{\mathbf{U}_{c}}+\underline{\mathbf{I}_{c}} \cdot\left(Z_{T L}+Z_{Y}\right)=0
\end{aligned}
$$

## One phase equivalent of a three-phase circuit

- We represent the circuit with a one-phase equivalent.
- As the system is balanced the electric magnitudes of the three phases have the same amplitude and a known phase shift (120 ${ }^{\circ}$ ).
- The behaviour of the whole system could be derived from the analysis of the so called one-phase equivalent or phase-neutral equivalent of the system.



## Analysis of circuits with delta-connected elements

- In systems with a delta-connected load or a delta-connected generator, it is not possible to apply the one-phase equivalent approach directly, since there is no neutral point.
- However it is possible to apply a wye delta transformation to obtain a YY connected system equivalent to the original.
- In $Y \Delta$ or $\Delta \Delta$ or $\Delta Y$ systems:

1. The system is transformed into a YY equivalent system.
2. The one-phase equivalent approach is applied over the YY equivalent circuit.

## $\Delta Y$ transformation for delta-connected generators

$\Delta$ connected generators can be redrawn as equivalent Y connected generators:


The two configurations are equivalent if:

$$
\underline{\mathbf{U}}_{L Y}=\underline{\mathbf{U}}_{L \Delta}
$$

## $\Delta Y$ transformation for three delta-connected impedances

We want to find the value of $Z_{Y}$ that makes the two loads equivalent


The loads are equivalent if for the same applied line voltages the line currents are the same (i.e: $\underline{\mathbf{U}}_{A B \Delta}=\underline{\mathbf{U}}_{A B Y}=>\underline{\mathbf{I}}_{A \Delta}=\underline{\mathbf{I}}_{A Y}$ )

## $\Delta Y$ transformation for three delta-connected impedances

The behaviour in the three phases is the same except that there is a phase shift of $120^{\circ}$. We analyse phase A:

$$
\begin{gathered}
\underline{\mathbf{I}}_{A \Delta}=\sqrt{3} \cdot \underline{\mathbf{I}}_{P h A} \angle-30^{\circ}=\frac{\underline{\mathbf{U}}_{A B} \cdot \sqrt{3} \angle-30^{\circ}}{Z_{\Delta}} \\
\underline{\mathbf{I}}_{A Y}=\underline{\mathbf{I}}_{P h A}=\frac{\underline{\mathbf{U}}_{A B} / \sqrt{3} \angle 30^{\circ}}{Z_{Y}}
\end{gathered}
$$

What value of impedance $Z_{Y}$ verifies the identity $\underline{\mathbf{I}}_{A \Delta}=\underline{\mathbf{I}}_{A Y}$ ?:

$$
\frac{\underline{\mathbf{u}}_{A B} \cdot \sqrt{3} \angle-30^{\circ}}{Z_{\Delta}}=\frac{\mathbf{U}_{A B} / \sqrt{3} \angle 30^{\circ}}{Z_{Y}}=>\quad Z_{Y}=\frac{Z_{\Delta}}{3}
$$

## Instantaneous power

The instantaneous power of three phase systems is constant*:

$$
p(t)=u_{a}(t) \cdot i_{a}(t)+u_{b}(t) \cdot i_{b}(t)+u_{c}(t) \cdot i_{c}(t)=3 \cdot U \cdot I \cdot \cos \varphi
$$

As power is constant the vibrations in the axles of three phase motors and generators are smaller that those in one-phase devices what makes them more stable from the mechanical point of view.

$$
\begin{array}{cc}
u_{a}(t)=\sqrt{2} \cdot U \cdot \cos (\omega t) & i_{a}(t)=\sqrt{2} \cdot l \cdot \cos (\omega t-\varphi) \\
u_{b}(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t-120^{\circ}\right) & i_{b}(t)=\sqrt{2} \cdot l \cdot \cos \left(\omega t-120^{\circ}-\varphi\right) \\
u_{c}(t)=\sqrt{2} \cdot U \cdot \cos \left(\omega t+120^{\circ}\right) & i_{c}(t)=\sqrt{2} \cdot l \cdot \cos \left(\omega t+120^{\circ}-\varphi\right)
\end{array}
$$

*Read the demonstration in the long notes

Active and reactive power of a three-phase wye load


$$
Z_{Y}=\left|Z_{Y}\right| \angle \theta
$$

$Q=3 \cdot U_{P h} \cdot I_{P h} \cdot \sin \varphi=\sqrt{3} \cdot U_{L} \cdot I_{L} \cdot \sin \varphi$

## Active and reactive power of a three-phase delta load

$$
\begin{gathered}
P=P_{A}+P_{B}+P_{C} \\
P_{A}=P_{B}=P_{C}=U_{P h} \cdot I_{P h} \cdot \cos \varphi \\
Z_{\Delta}=\left|Z_{\Delta}\right| \angle \theta \quad U_{L}=U_{P h} \quad I_{P h}=I_{L} / \sqrt{3}
\end{gathered}
$$

## Complex power of three-phase generators


$\mathcal{S}_{g}=\underline{\mathbf{U}}_{P h A} \cdot \underline{\mathbf{I}}_{P h A}^{*}+\underline{\mathbf{U}}_{P h B} \cdot \underline{\mathbf{I}}_{P h B}^{*}+\underline{\mathbf{U}}_{P h C} \cdot \underline{\mathbf{I}}_{P h C}^{*}=3 \cdot \underline{\mathbf{U}}_{P h A} \cdot \underline{\mathbf{I}}_{P h A}^{*}$

## Power factor

Power factor: $\quad$ p.f. $=\cos \varphi$


$$
\varphi=\varphi_{u}-\varphi_{i}=\arctan \frac{Q}{P}=\arctan \frac{X}{R}
$$

Resistive loads $\begin{array}{lllll}\mathbf{0} & 0 & 1\end{array}$
Inductive loads $>0>0 \quad 0<p . f<1 \quad$ lagging or inductive
Capacitive loads $<0<0 \quad 0<p . f<1$ leading or capacitive

## Reactive power compensation

- Many real-life loads, as electric motors, are highly inductive and often operation of power systems electric systems involves high amounts of reactive power transferred from the generators towards the loads.
- Fluctuating power increases the current flowing through the lines increasing losses and giving rise to voltage drops.
- Electric companies penalize the costumers that consume power with poor power factor.



## Reactive power compensation

Banks of capacitors are connected in parallel with the loads, to compensate part of the reactive power absorbed by them.


## Reactive power compensation

- Capacitors do not absorb or deliver any active power, so the active power of the system remains unchanged. $Q_{C}<0$
- The relation between the active an reactive power changes and the angle $\varphi^{\prime}$ becomes smaller.
- The power factor becomes closer to 1

Initial system
System with capacitors



$$
Q^{\prime}=Q+Q_{C}
$$

## Reactive power of a capacitor

A capacitor of capacitance $C$, a voltage $\operatorname{drop} \underline{\mathbf{U}}=U \angle \varphi_{u}$ and current flow $\underline{\mathbf{I}}=I \angle \varphi_{i}$

$$
\underbrace{\underline{\mathbf{I}}+\stackrel{\underline{\mathbf{U}}}{ }_{-}}_{\mathrm{Z}_{\mathrm{C}}=-\mathrm{j} / \omega \mathrm{C}}
$$

$$
Q_{C}=X_{C} \cdot I^{2}=\frac{U^{2}}{X_{C}}=-\omega \cdot C \cdot U^{2}
$$

$$
P_{C}=0
$$

## Reactive power of a bank of capacitors in wye



$$
Q_{C Y}=-3 \cdot \omega \cdot C_{Y} \cdot \underbrace{U_{P h}^{2}}_{U_{L} / \sqrt{3}}=-\omega \cdot C_{Y} \cdot U_{L}^{2}
$$

## Reactive power of a bank of capacitors in delta



$$
\begin{equation*}
Q_{C \Delta}=-3 \cdot \omega \cdot C_{\Delta} \cdot \underbrace{U_{P h}^{2}}_{U_{L}}=-3 \cdot \omega \cdot C_{\Delta} \cdot U_{L}^{2} \tag{1}
\end{equation*}
$$

## Capacitance required to get a target power factor

We want to compensate the reactive power of a system working with power factor $\cos \varphi$ so that the power factor becomes $\cos \varphi^{\prime}$

$Q=P \cdot \tan \varphi \quad Q^{\prime}=P \cdot \tan \varphi^{\prime} \quad Q_{C}=Q-Q^{\prime}=P \cdot\left(\tan \varphi-\tan \varphi^{\prime}\right)$

$$
C_{\Delta}=\frac{P \cdot\left(\tan \varphi-\tan \varphi^{\prime}\right)}{3 \cdot \omega \cdot U_{L}^{2}} \quad C_{Y}=\frac{P \cdot\left(\tan \varphi-\tan \varphi^{\prime}\right)}{\omega \cdot U_{L}^{2}}
$$

## Measure of power: working principle of wattmeters

A wattmeter is a measuring device that provides information on the power absorbed by electric dipoles.

Wattmeters incorporate two measuring circuits: the current coil and the voltage coil.

A sign * marks the terminals of the current and voltage coils of the same polarity


$$
W=I_{A} \cdot U_{A B} \cdot \cos \left(\widehat{\underline{U}_{A B} \underline{I}_{A}}\right)
$$

Measure of the active power in systems with accessible neutral point


$$
W=I_{A} \cdot U_{A N} \cdot \cos \left(\widehat{\underline{U}_{A N} \underline{I}_{A}}\right)=U_{P h} \cdot I_{P h} \cdot \cos \varphi=\frac{P}{3}
$$

Measure of the reactive power of a three-phase system with one wattmeter


Angle $\widehat{\underline{U}_{B C} \underline{I}_{A}}$


Measure of the reactive power of a three-phase system with one wattmeter


$$
W=U_{L} \cdot I_{L} \cdot \cos \left(\widehat{\mathbf{U}}_{B C} \underline{\underline{I}}_{A}\right)=U_{L} \cdot I_{A} \cdot \cos (90-\varphi)=U_{L} \cdot I_{A} \cdot \sin (\varphi)=\frac{Q}{\sqrt{3}}
$$

The two wattmeters method


$$
W_{1}=U_{A C} \cdot I_{A} \cdot \cos \left(\widehat{\widehat{\mathbf{U}}_{A C} \underline{\mathbf{I}}_{A}}\right)
$$

$$
W_{2}=U_{B C} \cdot I_{B} \cdot \cos \left(\widehat{\mathbf{U}_{B C} \underline{I}_{B}}\right)
$$

## Angles $\widehat{\underline{\mathbf{U}}_{A C} \underline{I}_{A}}$ and $\widehat{\underline{\mathrm{U}}_{B C} \underline{\mathbf{I}}_{B}}$



## The two wattmeters method

$$
\begin{aligned}
& W_{1}=U_{L} \cdot I_{L} \cdot \cos \left(\widehat{\widehat{U}_{A C} \underline{I}_{A}}\right)=U_{L} \cdot I_{L} \cdot \cos (30-\varphi)=U_{L} \cdot I_{L} \cdot\left(\frac{\sqrt{3}}{2} \cos \varphi+\frac{1}{2} \cdot \sin \varphi\right) \\
& W_{2}=U_{L} \cdot I_{L} \cdot \cos \left(\widehat{\underline{U}_{B C} \underline{I_{B}}}\right)=U_{L} \cdot I_{L} \cdot \cos (30+\varphi)=U_{L} \cdot I_{L} \cdot\left(\frac{\sqrt{3}}{2} \cos \varphi-\frac{1}{2} \cdot \sin \varphi\right)
\end{aligned}
$$

The active and reactive power of the three phase system can be obtained as the sum and the difference of the measures of the two wattmeters.

$$
\begin{gathered}
W_{1}+W_{2}=\sqrt{3} \cdot U_{L} \cdot I_{L} \cdot \cos \varphi=P \\
W_{1}-W_{2}=\cdot U_{L} \cdot I_{L} \cdot \sin \varphi=\frac{Q}{\sqrt{3}}
\end{gathered}
$$

