uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.1: Complex numbers

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1.1. COMPLEX NUMBERS

The set of *complex numbers* is defined as

$$\mathbf{C} = \{x + iy : x, y \in \mathbf{R}\},$$

where $i = \sqrt{-1}$. If $z = x + iy$, we say that x is the *real part* of z and y is the *imaginary part* of z, and we denote them by $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$. Note that the real part and the imaginary part are real numbers, and that real numbers are also complex numbers (in fact, they are the complex numbers with imaginary part 0). The complex numbers with real part 0 are called *pure imaginary numbers.* The only real number which is also a pure imaginary number is 0.

If we identify the complex number $x+iy$ with the ordered pair (x, y), we can represent the set of complex numbers as the Euclidean plane $\mathbf{R}^2$. We will call *real axis* to the horizontal axis and *imaginary axis* to the vertical axis. So, we will talk of the *complex plane* and its polar coordinates $r = |z|$, $\theta = \arg z$, the *modulus* and the *argument*, related with $z = x + iy \neq 0$ by

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x},$$
$$x = r\cos\theta, \qquad y = r\sin\theta.$$

It is well known that the argument θ is not well defined, since θ, $\theta + 2\pi$, $\theta + 4\pi$ and in general $\theta + 2k\pi$, with k any integer, are the same angle. Therefore, we must choose an argument; usual choices are $\theta \in [0, 2\pi)$ or $\theta \in (-\pi, \pi]$. It is clear that it does not exist a continuous choice of the function $\arg z$ on $\mathbf{C}\setminus\{0\}$. However, if S is any fixed half-line in the complex plane starting at 0, then there exists a continuous choice of the function $\arg z$ on $\mathbf{C} \setminus S$.

Definition. *The sum and the product of the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are defined, respectively, as*

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

The geometric interpretation of the sum of complex numbers is simple: if we identify a complex number z with the vector in $\mathbf{R}^2$ starting at 0 and ending at z, the complex number $z_1 + z_2$ is the endpoint of the sum of the vectors z_1 and z_2.

The geometric interpretation of the product of complex numbers is a little more complicated, since z_1z_2 is the complex number whose modulus is the product of the moduli of z_1 and z_2, and whose argument is the sum of the arguments of z_1 and z_2: if $z_1 = r_1\cos\theta_1 + i\,r_1\sin\theta_1$ and $z_2 = r_2\cos\theta_2 + i\,r_2\sin\theta_2$, then

$$\begin{aligned} z_1z_2 &= (r_1\cos\theta_1 + i\,r_1\sin\theta_1)(r_2\cos\theta_2 + i\,r_2\sin\theta_2) \\ &= r_1r_2\cos\theta_1\cos\theta_2 + i\,r_1r_2\cos\theta_1\sin\theta_2 + i\,r_1r_2\sin\theta_1\cos\theta_2 + i^2r_1r_2\sin\theta_1\sin\theta_2 \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\,r_1r_2(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= r_1r_2\cos(\theta_1 + \theta_2) + i\,r_1r_2\sin(\theta_1 + \theta_2). \end{aligned}$$

Definition. *The complex conjugate $\overline{z}$ of the complex number $z = x + iy$ is $\overline{z} = x - iy$.*

From the geometric viewpoint, $\overline{z}$ and z are symmetric with respect to the real axis. It is clear that $|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\,\overline{z}$. Thus, the inverse of $z \neq 0$ with respect to the multiplication is

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\,\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

One can check that the set of complex numbers with the sum and the product is a field.
The following properties hold:

$$|z_1 + z_2| \le |z_1| + |z_2| \,, \quad \big||z_1| - |z_2|\big| \le |z_1 - z_2| \,, \quad |\mathrm{Re}\, z| \le |z| \,, \quad |\mathrm{Im}\, z| \le |z| \,,$$
$$\mathrm{Re}\, z = \frac{z + \overline{z}}{2} \,, \qquad \mathrm{Im}\, z = \frac{z - \overline{z}}{2i} \,,$$
$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2 \,, \quad \overline{z_1 z_2} = \overline{z}_1 \overline{z}_2 \,, \quad \overline{(z^n)} = \overline{z}^n \,, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2} \,,$$
$$|\overline{z}| = |z| \,, \quad \arg \overline{z} = -\arg z \,,$$
$$|z_1 z_2| = |z_1|\,|z_2| \,, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \,, \quad |z^n| = |z|^n \,, \quad \left|\sqrt[n]{z}\,\right| = \sqrt[n]{|z|} \,.$$

Since if θ is an argument of a complex number, then $\theta + 2k\pi$ is also an argument for any $k \in \mathbf{Z}$, the following formulas hold in two senses: *a*) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ means that for any choice of $\arg z_1$ and $\arg z_2$, we have that $\arg z_1 + \arg z_2$ is an argument of $z_1 z_2$; *b*) the formulas are equalities of sets if we consider $\arg z$ as the set of every argument of z.

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \,, \quad \arg\left(\frac{1}{z}\right) = \arg \overline{z} = -\arg z \,,$$
$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \,, \quad \arg(z^n) = n \arg z \,,$$
$$\arg(\sqrt[n]{z}\,) = \frac{\arg z}{n} = \left\{ \frac{\theta + 2k\pi}{n} \;:\; \theta \in \arg z \,,\ k \in \mathbf{Z} \right\}.$$

Also, the De Moivre's formula $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ holds. If we define $e^{i\theta} = \cos\theta + i\sin\theta$, then this formula can be written as $(e^{i\theta})^n = e^{in\theta}$ and so, this is an interesting notation.

Thus, we can compute roots of order n of complex numbers in a simple way, since θ and $\theta + 2k\pi$ are the same angle for any $k \in \mathbf{Z}$:

$$\sqrt[n]{z} = \sqrt[n]{r\, e^{i\theta}} = \sqrt[n]{r\, e^{i(\theta + 2k\pi)}} = \sqrt[n]{r}\, e^{i(\theta+2k\pi)/n} \,, \qquad k \in \mathbf{Z} \,.$$

By taking the values $k = 0, 1, \ldots, n-1$ we obtain the n-th roots of z

$$\sqrt[n]{r}\, e^{i\theta/n} \,, \sqrt[n]{r}\, e^{i(\theta+2\pi)/n} \,, \ldots \,, \sqrt[n]{r}\, e^{i(\theta+2(n-1)\pi)/n} \,,$$

and one can check that we obtain one of these n complex numbers for any value of $k \in \mathbf{Z}$. In particular, any real number has n complex n-th roots. For instance, if we consider the case $n = 2$, we obtain two square roots

$$\left\{\sqrt{r}\, e^{i\theta/2} \,, \sqrt{r}\, e^{i(\theta+2\pi)/2}\right\} = \left\{\sqrt{r}\, e^{i\theta/2} \,, \sqrt{r}\, e^{i\theta/2 + i\pi}\right\} = \pm\sqrt{r}\, e^{i\theta/2} \,.$$

If $z = x + iy = re^{i\theta}$, we say that $x + iy$ is the *binomial form* of z and $re^{i\theta}$ is the *exponential form* of z.