uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.3: Power series

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

1.3. POWER SERIES

Definitions and previous results.

1. If $\omega_n = a_n + ib_n$, $\omega = a + ib$,

$$\lim_{n\to\infty} \omega_n = \omega \quad \Leftrightarrow \quad \forall \varepsilon > 0\,, \ \exists N \ \text{ such that } \ |\omega_n - \omega| < \varepsilon\,, \ \forall n \geq N$$
$$\Leftrightarrow \quad \lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b\,.$$

2. We say that $\lim_{n\to\infty} \omega_n = \infty$ if $\lim_{n\to\infty} |\omega_n| = \infty$.

3. We say that the sequence $\{\omega_n\}$ is <u>convergent</u> if it has a finite limit.

EXAMPLES. $e^{-n} + i(n+1)/n$, $(-1)^n n$, i^n.

4. The series $\sum_{n=1}^\infty \omega_n$ is, by definition, the limit $\lim_{N\to\infty} \sum_{n=1}^N \omega_n$, and we say that it is convergent if this limit is finite.

5. If $\sum_{n=1}^\infty \omega_n$ converges, then $\lim_{n\to\infty} \omega_n = 0$, i.e., if $\lim_{n\to\infty} \omega_n$ is different from 0 or it does not exists, then the series <u>diverges</u>. The converse does not hold:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{and} \qquad \sum_{n=1}^\infty \frac{1}{n} = \infty.$$

6. The series $\sum_{n=1}^\infty \omega_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^\infty |\omega_n|$ is convergent.

7. If $\sum_{n=1}^\infty |\omega_n|$ converges, then $\sum_{n=1}^\infty \omega_n$ converges: If $\sum_{n=1}^\infty |\omega_n|$ converges, then $\sum_{n=1}^k |\omega_n|$ is a Cauchy sequence and for each $\varepsilon > 0$ there exists n_0 such that $\sum_{n=M}^N |\omega_n| < \varepsilon$ for every $M, N \geq n_0$. Thus,

$$\Big| \sum_{n=M}^N \omega_n \Big| \leq \sum_{n=M}^N |\omega_n| < \varepsilon\,,$$

$\sum_{n=1}^k \omega_n$ is a Cauchy sequence and so, $\sum_{n=1}^\infty \omega_n$ converges.

The converse does not hold:

$$\sum_{n=1}^\infty \frac{(-1)^n}{n} \ \text{ converges} \qquad \text{but} \qquad \sum_{n=1}^\infty \frac{1}{n} = \infty.$$

8. Consider $a_n \geq 0$.

(1) If there exists $\lim_{n\to\infty} a_n^{1/n} = r$, then

$$\begin{cases} r > 1 \implies \displaystyle\sum_{n=1}^\infty a_n \ \text{ diverges,} \\ 0 \leq r < 1 \implies \displaystyle\sum_{n=1}^\infty a_n \ \text{ converges.} \end{cases}$$

(2) If $a_n > 0$ and there exists $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$, then $r = \lim_{n\to\infty} a_n^{1/n}$ and the same as in (1) is concluded.

(3) If $r = 1$ in (1) or (2), then the series can diverge or converge: $\sum_{n=1}^\infty 1/n$ diverges and $\sum_{n=1}^\infty 1/n^2$ converges.

EXERCISE. $\sum_{n=1}^\infty \frac{1+i2^n}{3^n+in}$.

9. Consider $f_n : \Omega \longrightarrow \mathbf{C}$. We say that $\{f_n\}$ <u>converges pointwisely</u> to f on Ω, if

$$\lim_{n\to\infty} f_n(z) = f(z)\,, \quad \forall\, z \in \Omega\,,$$

i.e., if

$$\forall\, z \in \Omega\,,\ \forall\, \varepsilon > 0\,,\ \exists\, N = N(z,\varepsilon) \ \text{ such that } \ |f_n(z) - f(z)| < \varepsilon\,, \quad \forall\, n \geq N\,.$$

10. Consider $f_n : \Omega \longrightarrow \mathbf{C}$. We say that $\{f_n\}$ <u>converges uniformly</u> to f on Ω, if

$$\forall\, \varepsilon > 0\,,\ \exists\, N = N(\varepsilon) \ \text{ such that } \ |f_n(z) - f(z)| < \varepsilon\,, \quad \forall\, n \geq N\,, \quad \forall z \in \Omega\,.$$

11. Uniform convergence implies pointwise convergence. The converse does not hold: The sequence

$$f_n(x) = x^n$$

converges pointwisely to the function $f(x) = 0$ on the interval $[0,1)$, but it does not converge uniformly to $f(x) = 0$ on $[0,1)$.

12. If f_n converges uniformly to f on Ω, and f_n is a continuous function on Ω for each n, then f is also a continuous function on Ω.

13. Weierstrass M test: If

$$|f_n(z)| \leq M_n\,,\ \forall\, n \geq n_0\,,\ \forall z \in \Omega \quad \text{and} \quad \sum_{n=n_0}^{\infty} M_n < \infty\,,$$

then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on Ω.

Recall that if $c > 0$, then c^x is defined as $c^x = e^{x \log c}$, and so,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y),$$
$$c^z = e^{z \log c} = e^{x \log c}\big(\cos(y \log c) + i \sin(y \log c)\big),$$
$$\left|c^z\right| = e^{x \log c} = c^x = c^{\operatorname{Re} z}.$$

Example. The <u>Riemann Zeta function</u> $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely and uniformly on $\{z \in \mathbf{C} : \operatorname{Re} z \geq a\}$ if $a > 1$. We have $n^{\operatorname{Re} z} \geq n^a$,

$$\left|\frac{1}{n^z}\right| = \frac{1}{n^{\operatorname{Re} z}} \leq \frac{1}{n^a}\,, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^a} \ \text{ converges since } a > 1.$$

14. Given a sequence $\{a_n\} \subset \mathbf{R}$, the <u>upper limit</u> of $\{a_n\}$ is defined as

$$\limsup_{n\to\infty} a_n = \inf_{n\in\mathbf{N}} \Big(\sup_{k\geq n} a_k\Big) = \lim_{n\to\infty} \Big(\sup_{k\geq n} a_k\Big),$$

i.e., it is the supremum of the set given by the limits of subsequences of $\{a_n\}$. Hence, $\limsup_{n\to\infty} a_n$ **always exists** (although it can be infinite). Furthermore,

(a) $\exists \lim_{n\to\infty} a_n \implies \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(c) If $a_n \geq 0$ and there exists $\lim_{n\to\infty} a_n \implies \limsup_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$.

(d) If $a_n, b_n \geq 0$, $\limsup_{n\to\infty} a_n b_n \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$.

(e) If $a_n, b_n \geq 0$ and there exists $\lim_{n\to\infty} b_n > 0$, then $\limsup_{n\to\infty} a_n^{b_n} = (\limsup_{n\to\infty} a_n)^{\lim_{n\to\infty} b_n}$.

(f) If f is an increasing continuous function, then $\limsup_{n\to\infty} f(a_n) = f(\limsup_{n\to\infty} a_n)$.

EXAMPLES. $\limsup_{n\to\infty} \frac{n}{n+1} = 1$, $\limsup_{n\to\infty}(-1)^n = 1$, $\limsup_{n\to\infty} \frac{n}{n+i}$ has no sense.

Definition. *A power series centered at z_0 (or about z_0) is a series as*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \,. \tag{1}$$

As usual, $(z-z_0)^0$ evaluates as 1 and the sum of the series is thus a_0 for $z = z_0$.

Theorem 1. *The series* (1)

(a) *converges absolutely on $D(z_0, R) = \{z : |z-z_0| < R\}$, where R is given by the Cauchy-Hadamard formula*

$$0 \le R := \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} \le \infty \,,$$

with absolute and uniform convergence on each closed disk $\overline{D(z_0,r)} \subset D(z_0,R)$, with $r < R$;

(b) *diverges for each z such that $|z-z_0| > R$.*

R is called the radius of convergence of the series and the disk $D(z_0,R)$ is called the disk of convergence of the series.

PROOF.

(a) We can assume that $R > 0$. Let us choose $0 < r < \rho < R$. Since

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} < \frac{1}{\rho} \,,$$

there exists N such that $|a_n| < 1/\rho^n$ for every $n \ge N$, and so, if $|z-z_0| \le r$, then

$$|a_n(z-z_0)^n| = |a_n|\,|z-z_0|^n < \frac{1}{\rho^n}\, r^n = \left(\frac{r}{\rho}\right)^n, \qquad n \ge N \,.$$

Since the series $\sum_{n=N}^{\infty}(r/\rho)^n$ is convergent, the Weierstrass M test gives the result.

(b) We can assume that $R < \infty$. Assume that $|z-z_0| = r$ with $r > R$. Since

$$\limsup_{n\to\infty} |a_n|^{1/n} = \frac{1}{R} > \frac{1}{r} \,,$$

there exists a subsequence $\{a_{n_k}\}$ such that $|a_{n_k}| > 1/r^{n_k}$ for every k, and so,

$$|a_{n_k}(z-z_0)^{n_k}| = |a_{n_k}|\,|z-z_0|^{n_k} > \frac{1}{r^{n_k}}\, r^{n_k} = 1 \,,$$

for every k. Thus, we do not have $\lim_{n\to\infty} a_n(z-z_0)^n = 0$ and so, the series (1) diverges. $\sharp$

EXAMPLE. We will need the following limit

$$\lim_{n\to\infty} (\alpha\, n^{\beta})^{\gamma/n} = 1$$

for every $\alpha > 0, \beta, \gamma \in \mathbf{R}$. Denote by L the limit $L = \lim_{n\to\infty}(\alpha\, n^{\beta})^{\gamma/n}$. Thus,

$$\log L = \log\big(\lim_{n\to\infty}(\alpha\, n^{\beta})^{\gamma/n}\big) = \lim_{n\to\infty}\log\big((\alpha\, n^{\beta})^{\gamma/n}\big) = \lim_{n\to\infty}\frac{\gamma\log(\alpha\, n^{\beta})}{n} = 0 \,,$$

and so, $L = e^0 = 1$.

EXERCISES. $\sum_{n=1}^{\infty} n^n z^n$, $\sum_{n=1}^{\infty} n^{\alpha} z^n$ $(\alpha \in \mathbf{R})$, $\sum_{n=0}^{\infty} z^{n!}$.

Next theorem allows to compute the radius of convergence in many cases:

Theorem 2. *If there exists* $\lim_{n\to\infty} |a_{n+1}|/|a_n|$, *then the radius of convergence* R *of* (1) *satisfies*

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\,.$$

PROOF. Assume first that $R > 0$ and consider $z \in \mathbf{C}$, $r, r_0 > 0$ with $|z - z_0| \le r < r_0 < R$. Since

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R} < \frac{1}{r_0}\,,$$

there exists N such that $|a_{n+1}| < |a_n|/r_0$ for every $n \ge N$, and so, $|a_n| < |a_N|/r_0^{n-N}$ for every $n \ge N$. Since $|z - z_0| \le r$, we have for every $n \ge N$,

$$|a_n(z-z_0)^n| = |a_n|\,|z-z_0|^n < \frac{|a_N|}{r_0^{n-N}}\, r^n = |a_N|\, r_0^N \left(\frac{r}{r_0}\right)^n.$$

Since the series $\sum_{n=N}^{\infty} (r/r_0)^n$ is convergent, the Weierstrass M test gives the first part of the result.

Assume now that $R < \infty$ and consider $z \in \mathbf{C}$, $\rho > 0$ with $|z - z_0| = \rho > R$. Since

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R} > \frac{1}{\rho}\,,$$

there exists N such that $|a_{n+1}| > |a_n|/\rho$ for every $n \ge N$, and so, $|a_n| > |a_N|/\rho^{n-N}$ for every $n \ge N$. Hence,

$$|a_n(z-z_0)^n| = |a_n|\,|z-z_0|^n > \frac{|a_N|}{\rho^{n-N}}\, \rho^n = |a_N|\, \rho^N\,,$$

for every n. Thus, we do not have $\lim_{n\to\infty} a_n (z-z_0)^n = 0$ and so, the series diverges. ♯

EXERCISES. $\sum_{n=0}^{\infty} z^n/n!$, $\sum_{n=1}^{\infty} n^{\alpha} z^n$ $(\alpha \in \mathbf{R})$, $\sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!$.

We say that two sequences $\{a_n\}, \{b_n\}$, are comparable if there exist constants $c_1, c_2, N > 0$, such that $c_1 \le a_n/b_n \le c_2$ for every $n \ge N$. If the coefficients of a series are comparable with n^{α} (for some $\alpha \in \mathbf{R}$), then the radius of convergence of the series is 1.

Theorem 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ *and assume that the radius of convergence* R *of the series is strictly positive. Then* f *is holomorphic on* $D(z_0, R)$, *and*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}\,, \qquad \forall\, z \in D(z_0, R)\,.$$

Furthermore, the radius of convergence of the series of f' *is also* R. *Also,* R *is the radius of convergence of the series*

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}\,(z-z_0)^{n+1}\,.$$

PROOF. Let us consider the function $g(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n$, whose radius of convergence R_g is

$$\frac{1}{R_g} = \limsup_{n\to\infty} |(n+1)a_{n+1}|^{1/n} = \lim_{n\to\infty} (n+1)^{1/n} \limsup_{n\to\infty} \left(|a_{n+1}|^{1/(n+1)}\right)^{(n+1)/n} = 1 \cdot \left(\frac{1}{R}\right)^1 = \frac{1}{R}\,.$$

Hence, $R_g = R$ and for each $0 < r < R$, Theorem 1 gives that the series which define f and g converge uniformly on $\overline{D(z_0, r)}$. Consequently, $g = f'$ on $D(z_0, R)$. The same argument gives the result for the other series. ♯

Corollary. *If* $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $\forall\, z \in D(z_0, R)$, *then*

(1) *f is differentiable infinite times on $D(z_0, R)$, and also*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}\,, \qquad \forall\, k \in \mathbf{N}\,,\ \forall\, z \in D(z_0, R)\,.$$

(2) $a_k = f^{(k)}(z_0)/k!$, $\forall\, k \in \mathbf{N}$.

(3) *If two power series are equal at every point of a disk $D(z_0, \varepsilon)$ for some $\varepsilon > 0$, then they are equal:*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n\,, \quad \forall\, z \in D(z_0, \varepsilon) \implies a_n = b_n\,,\ \forall\, n \in \mathbf{N}\,.$$

Theorem 1 guarantees the convergence of (1) on the interior of the disk of convergence. The study of the convergence on the boundary of the disk it is much more difficult, since, in general, the series may converge for some values of z and diverge for others. The following result is useful.

Theorem 4. *Let R be the radius of convergence of* (1). *If* $a_0 \geq a_1 R \geq a_2 R^2 \geq \cdots$ *and* $\lim_{n\to\infty} a_n R^n = 0$, *then* $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ *converges on the circumference* $\{|z - z_0| = R\}$ *unless, perhaps, at the point* $z = z_0 + R$.

Proof. Let us recall the *Dirichlet criterion*: If $\{b_n\} \subset \mathbf{C}$ and its partial sums are a bounded sequence, $c_0 \geq c_1 \geq \cdots$, and $\lim_{n\to\infty} c_n = 0$, then the series $\sum_{n=0}^{\infty} b_n c_n$ is convergent.

Assume that $z = z_0 + Re^{i\theta}$ with $\theta \in (0, 2\pi)$. Thus,

$$\frac{(z - z_0)^n}{R^n} = \frac{\left(Re^{i\theta}\right)^n}{R^n} = e^{i\theta n}\,,$$

and

$$\Big|\sum_{n=0}^{N} e^{i\theta n}\Big| = \Big|\frac{1 - e^{i\theta(n+1)}}{1 - e^{i\theta}}\Big| \leq \frac{2}{|1 - e^{i\theta}|}\,.$$

Hence, if $z = z_0 + Re^{i\theta}$ and $\theta \in (0, 2\pi)$, Dirichlet criterion with $b_n = e^{i\theta n}$ and $c_n = a_n R^n$, applied to the series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n R^n \frac{(z - z_0)^n}{R^n} = \sum_{n=0}^{\infty} a_n R^n e^{i\theta n},$$

gives the result. ♯

Examples.

$\sum_{n=1}^{\infty} \frac{z^n}{n+1}$ converges on the circumference $\{|z| = 1\}$ unless at the point $z = 1$.

$\sum_{n=1}^{\infty} (-1)^n \frac{(z-3)^n}{\sqrt{n}}$ converges on the circumference $\{|z - 3| = 1\}$ unless at the point $z = 2$.

Definition. *A function is analytic at z_0 if it can be written as a power series centered at z_0. A function is analytic on a set A if it is analytic at every point in A.*

Why does converges the real series of the function $f(x) = \frac{1}{1+x^2} = \sum_{n=1}^{\infty} (-1)^n x^{2n}$ just on $\{|x| < 1\}$, when f is a function of class C^∞ (and in fact analytic) at every point in the real axis?

When is a real function real analytic?

These are main questions in real analysis. However, there is no answer to these questions from a real analysis viewpoint.