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Departamento de Matemáticas

Complex variable and transforms

Chapter 1: Complex variable

Section 1.4: Elementary functions

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1.4. ELEMENTARY FUNCTIONS

Exponential function. We define the exponential function as the power series (which has infinite radius of convergence)

$$e^z := f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbf{C}. \tag{2}$$

With this definition it is easy to prove the main properties of the exponential function:

1. $f(x) = e^x$ for every $x \in \mathbf{R}$, i.e. $f(x)$ is the real exponential function. In fact, using the real Taylor's Theorem we have

$$e^x - \sum_{n=0}^{N} \frac{x^n}{n!} = e^{\xi} \frac{x^{N+1}}{(N+1)!} \equiv E_N(x), \qquad 0 < \xi < x, \ (\text{or } x < \xi < 0),$$

and the property is a consequence of

$$|E_N(x)| \le e^{|x|} \frac{|x|^{N+1}}{(N+1)!} \underset{N\to\infty}{\longrightarrow} 0.$$

2. $f(z)\,f(w) = f(z+w)$, $\forall\, z, w \in \mathbf{C}$. In fact:

$$\begin{aligned} f(z)\,f(w) &= \Big(\sum_{k=0}^{\infty} \frac{z^k}{k!}\Big)\Big(\sum_{m=0}^{\infty} \frac{w^m}{m!}\Big) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^k w^m}{k!m!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{z^j}{j!}\,\frac{w^{n-j}}{(n-j)!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j} \\ &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = f(z+w). \end{aligned}$$

3. If $y \in \mathbf{R}$, then $f(iy) = \cos y + i \sin y$, since:

$$\begin{aligned} \cos y &= \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!}, \\ i \sin y &= i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!}, \end{aligned}$$

and summing, we obtain

$$\cos y + i \sin y = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = f(iy).$$

4. If $z = x + iy \in \mathbf{C}$, then $f(x+iy) = f(x)\,f(iy) = e^x(\cos y + i \sin y)$, and this justify the definition of (2) as the exponential function.
5. The exponential function has period $2\pi i$, since $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z$.
6. Computing the term by term derivative of the series (2), we obtain $(e^z)' = e^z$.
7. $|e^z| = e^{\operatorname{Re} z}$, $\arg e^z = \operatorname{Im} z$, since $e^{x+iy} = e^x(\cos y + i \sin y)$.
8. $e^{\bar z} = \overline{e^z}$.

As in the case of the exponential function, if a function g can be written as

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \text{with } a_n \in \mathbf{R} \quad \forall\, n \in \mathbf{N},$$

then $g(\bar{z}) = \overline{g(z)}$.

Logarithm function. We say that

$$w = \log z \iff z = e^w .$$

If $w = \log z = u + iv$ with $z = re^{i\theta}$, then we have

$$z = re^{i\theta} = e^w = e^u e^{iv} \implies \begin{cases} r = e^u , \\ \theta = v . \end{cases} \implies \begin{cases} u = \log r , \\ v = \theta . \end{cases}$$

Hence,

$$\log z = \log r + i\,\theta ,$$
$$\log z = \log|z| + i \arg z .$$

For $\log z$ to be a single-valued function (a true function) we need to choose an interval of arguments with length 2π. The principal value of the logarithm is obtained by taking $\arg z \in (-\pi, \pi]$.

Cauchy-Riemann equations give that $\log z$ is a holomorphic function on the domain where it is continuous. Thus, $\log z$ is holomorphic on $\mathbf{C} \setminus (-\infty, 0]$ if we consider the arguments in the interval $(-\pi, \pi)$.

It is easy to check that $(\log z)' = 1/z$, by computing the partial derivative of $\log z$ with respect to x, or by using that the logarithm is the inverse function of the exponential.

Note that

$$e^{\log z} = z ,$$

for every $z \in \mathbf{C}$, since by definition of logarithm,

$$w = \log z \iff z = e^w .$$

But it is possible to have

$$\log e^z \neq z ,$$

for some values of z: If we consider the principal value of the logarithm and $z = 2\pi i$, then

$$\log e^z = \log e^{2\pi i} = \log 1 = 0 \neq 2\pi i = z .$$

General exponential and potential functions. If $a \in \mathbf{C}$, the functions $f(z) = a^z$ and $g(z) = z^a$ are defined as

$$a^z = e^{z \log a} \ (a \neq 0) , \qquad z^a = e^{a \log z} .$$

We have

$$\big(a^z\big)' = a^z \log a , \qquad \big(z^a\big)' = a\, z^{a-1} .$$

Hence, a^z is holomorphic on $\mathbf{C}$, and z^a is holomorphic on $\mathbf{C} \setminus (-\infty, 0]$; if $a \in \mathbf{Z}$, z^a is holomorphic on $\mathbf{C} \setminus \{0\}$, and if $a \in \mathbf{N}$, z^a is holomorphic on $\mathbf{C}$.

Trigonometric functions. The functions sine and cosine are defined by the following power series with infinite radius of convergence:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} , \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} .$$

We have

$$\big(\cos z\big)' = -\sin z , \qquad \big(\sin z\big)' = \cos z .$$

By using the arguments in the proofs of the properties of the exponential function, we have

$$e^{iz} = \cos z + i \sin z\,, \qquad \forall z \in \mathbf{C}\,. \tag{3}$$

By definition of $\sin z$ and $\cos z$ by power series, we have $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$, and so,

$$e^{-iz} = \cos z - i \sin z\,. \tag{4}$$

From (3) and (4) we obtain

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}\,, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}\,. \tag{5}$$

The other trigonometric functions can be defined in terms of sine and cosine:

$$\tan z = \frac{\sin z}{\cos z}\,, \qquad \operatorname{cotan} z = \frac{\cos z}{\sin z}\,, \qquad \sec z = \frac{1}{\cos z}\,, \qquad \operatorname{cosec} z = \frac{1}{\sin z}\,,$$

and they are holomorphic on $\mathbf{C}$ unless the zeros of the corresponding denominators.

Using the relations (5) it is easy to obtain (for complex values) many of the known relations for the real case, for example,

$$\cos^2 z + \sin^2 z = 1\,, \qquad \sin 2z = 2 \sin z \cos z\,, \qquad \cos 2z = \cos^2 z - \sin^2 z\,.$$

Hyperbolic functions. As usual, the hyperbolic functions are defined as

$$\cosh z = \frac{e^{z} + e^{-z}}{2}\,, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}\,, \tag{6}$$

which are holomorphic on $\mathbf{C}$, and

$$\tanh z = \frac{\sinh z}{\cosh z}\,, \qquad \operatorname{cotanh} z = \frac{\cosh z}{\sinh z}\,, \qquad \operatorname{sech} z = \frac{1}{\cosh z}\,, \qquad \operatorname{cosech} z = \frac{1}{\sinh z}\,,$$

which are holomorphic on $\mathbf{C}$ unless the zeros of the corresponding denominators. We have

$$\big(\cosh z\big)' = \sinh z\,, \qquad \big(\sinh z\big)' = \cosh z\,.$$

(5) and (6) provide the following relations with the trigonometric functions

$$\begin{aligned} \cos z &= \cosh(iz)\,, \qquad & \sin z &= -i \sinh(iz)\,, \\ \cosh z &= \cos(iz)\,, \qquad & \sinh z &= -i \sin(iz)\,. \end{aligned}$$

We also have the formulas

$$\cosh^2 z - \sinh^2 z = 1\,, \qquad \sinh 2z = 2\, \sinh z \cosh z\,, \qquad \cosh 2z = \cosh^2 z + \sinh^2 z\,.$$