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Departamento de Matemáticas

# Complex variable and transforms

## Chapter 1: Complex variable

### Section 1.6: Cauchy's integral formula

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## 1.6. CAUCHY'S INTEGRAL FORMULA

The following wonderful formula can be deduced from Cauchy's integral theorem with singularities:

**Theorem 1 (Cauchy's integral formula).** *If $\Omega$ is a simply connected open set and $f : \Omega \longrightarrow \mathbf{C}$ is holomorphic, then*

$$n(\gamma, a)\, f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz\,,$$

*for every point $a \in \Omega$, and for every closed curve $\gamma \subset \Omega \setminus \{a\}$.*

Proof. Fix $a \in \Omega$. The function defined by

$$F(z) := \frac{f(z) - f(a)}{z-a}$$

is holomorphic on $\Omega \setminus \{a\}$ and $\lim_{z \to a} (z-a)\, F(z) = 0$ since $f$ is continuous on $\Omega$. Cauchy's integral theorem (with singularities) gives

$$\int_\gamma F(z)\, dz = 0\,, \qquad \text{for every closed curve } \gamma \subset \Omega \setminus \{a\}\,.$$

Therefore,

$$\int_\gamma \frac{f(z)}{z-a}\, dz = f(a) \int_\gamma \frac{dz}{z-a} = 2\pi i\, n(\gamma, a)\, f(a)\,. \quad \sharp$$

Recall that if $\gamma$ is a simple closed curve in the complex plane, the Jordan curve theorem says that the open set $\mathbf{C} \setminus \gamma$ has two connected components: $\operatorname{Ext}\gamma$ (containing a neighborhood of the point at infinity) and $\operatorname{Int}\gamma$ (which is simply connected). Cauchy's integral formula gives that the values on $\operatorname{Int}\gamma$ of any holomorphic function can be obtained from its values on $\gamma$.

Remark: Unless otherwise stated, we will assume that the simple closed curves are positively oriented, i.e., in the counterclockwise direction.

**Corollary 1.** *If $\gamma$ is a simple closed curve and $f$ is holomorphic on an open set containing $\gamma \cup Int\gamma$, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = \begin{cases} f(a)\,, & \textit{if } a \in \operatorname{Int}\gamma\,, \\ 0\,, & \textit{if } a \in \operatorname{Ext}\gamma\,. \end{cases}$$

In particular, if $\gamma$ is the boundary of a disk, the following holds:

**Corollary 2.** *If $f$ is holomorphic on the open set $\Omega$, and $D$ is a disk such that $\overline{D} \subset \Omega$, then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z}\, dw\,, \qquad \forall\, z \in D\,.$$

Cauchy's integral formula has a surprising consequence: holomorphic functions are infinitely differentiable and their derivatives are holomorphic functions.

**Theorem 2 (Cauchy's integral formula for the derivatives).** *If $f : \Omega \longrightarrow \mathbf{C}$ is holomorphic on the open set $\Omega$, then there exist all the derivatives $f^{(n)}$ of $f$, and they are holomorphic on $\Omega$. Furthermore, if $D$ is an open disk such that $\overline{D} \subset \Omega$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}}\, dw\,, \qquad \forall\, z \in D\,,\ \forall\, n \in \mathbf{N}\,.$$

The proof of Cauchy's integral formula for the derivatives uses the following result on differentiation under the integral sign.

**Theorem 3.** *Let $E \subseteq \mathbf{R}^N$, $\Omega \subseteq \mathbf{C}$ an open set, and $g : E \times \Omega \longrightarrow \mathbf{C}$ a continuous function satisfying:*

(1) *$g(x, \cdot) : \Omega \longrightarrow \mathbf{C}$ is holomorphic on $\Omega$ for each fixed $x \in E$.*

(2) *We have*

$$\Big| \frac{\partial g}{\partial z}(x,z) \Big| \leq h(x)\,,$$

*for every $x \in E$ and every $z \in \Omega$, where $h : E \longrightarrow [0, \infty]$ is an integrable function on $E$.*

*Then the function defined by*

$$f(z) = \int_E g(x,z)\,dx$$

*is holomorphic on $\Omega$, and*

$$f'(z) = \int_E \frac{\partial g}{\partial z}(x,z)\,dx\,.$$

Proof of Theorem 2. Since $f$ is a continuous function on the compact set $\overline{D}$, there exists a constant $M$ such that $|f(w)| \leq M$ for every $w \in \partial D$. Also,

$$\Big| \frac{1}{(w-z)^{n+1}} \Big| \leq \frac{1}{\mathrm{dist}\,(z, \partial D)^{n+1}}\,,$$

and so, for every $z \in \Omega$ and $w \in \partial D$,

$$\Big| \frac{f(w)}{(w-z)^{n+1}} \Big| \leq \frac{M}{\mathrm{dist}\,(z, \partial D)^{n+1}}\,.$$

For each $\varepsilon > 0$ consider the set $D_\varepsilon := \{z \in D / \,\mathrm{dist}\,(z, \partial D) \geq \varepsilon\}$; we have for every $z \in D_\varepsilon$ and $w \in \partial D$,

$$\Big| \frac{f(w)}{(w-z)^{n+1}} \Big| \leq \frac{M}{\varepsilon^{n+1}}\,.$$

Since $\partial D$ has finite length, $M/\varepsilon^{n+1}$ is integrable on $\partial D$ and it does not depend on $z$. Also, $f(w)/(w-z)^{n+1}$ is holomorphic on $D$ for each $w \in \partial D$. Thus, Theorem 2 follows from Theorem 8 by applying induction. ♯

In fact, the argument in the proof of Theorem 2 gives the following result for "Cauchy-type" integrals.

**Theorem 4.** *Let $\gamma : [a, b] \longrightarrow \mathbf{C}$ be a curve and $\varphi \in L^1(\gamma)$. Then,*

$$F(z) = \int_\gamma \frac{\varphi(w)}{w - z}\,dw$$

*is holomorphic on $\mathbf{C} \setminus \gamma$, and*

$$F^{(n)}(z) = n! \int_\gamma \frac{\varphi(w)}{(w-z)^{n+1}}\,dw\,, \qquad \forall\, z \in \mathbf{C} \setminus \gamma\,.$$

We are going to obtain many interesting consequences of Cauchy's integral formula. The first one is a partial converse of Cauchy's integral theorem.

**Theorem 5 (Morera's theorem).** *Let $\Omega \subseteq \mathbf{C}$ be an open set and $f : \Omega \longrightarrow \mathbf{C}$ a continuous function such that*

(1) $$\int_\gamma f = 0\,, \qquad \text{for every closed curve } \gamma : [a,b] \longrightarrow \Omega\,.$$

*Then $f$ is holomorphic on $\Omega$.*

Proof. It suffices to prove Morera's theorem on each connected component of $\Omega$. Let $A$ be a connected component of $\Omega$ and $z_0$ a point in $A$. (1) implies that the function

$$F(z) = \int_{z_0}^{z} f(w)\, dw\,, \qquad z \in A\,,$$

is well defined. The argument in the proof of Fundamental Theorem of Calculus gives that $F$ is differentiable on $A$ and $F' = f$. Hence, $F$ is holomorphic on $A$, and Theorem 2 gives that $f = F'$ is also holomorphic on $A$. ♯

**Corollary 3.** *Let $\sigma : [a,b] \longrightarrow \mathbf{C}$ be a curve and $\Omega \subseteq \mathbf{C}$ an open set. Let $f : \Omega \times \sigma([a,b]) \longrightarrow \mathbf{C}$ be a continuous function such that $f(\cdot, w) : \Omega \longrightarrow \mathbf{C}$ is holomorphic on $\Omega$ for each fixed $w \in \sigma([a,b])$. Assume that there exists a function $g \in L^1(\sigma)$ with $|f(z,w)| \leq g(w)$ for every $z \in \Omega$ and every $w \in \sigma([a,b])$. Then the function*

$$F(z) = \int_{\sigma} f(z,w)\, dw$$

*is holomorphic on $\Omega$.*

Proof. Exercise. *Hint*: Prove first that $F$ is a continuous function on $\Omega$, and check that $\int_\gamma F(z)\, dz = 0$ for any closed curve $\gamma$ contained in $\Omega$. ♯

**Theorem 6 (Cauchy's inequality).** *If $f$ is holomorphic on $\overline{D(a,r)} = \{z \in \mathbf{C} : \; |z-a| \leq r\}$ and*

$$M_r = \max\{|f(z)| : \; |z-a| = r\},$$

*then*

$$\left|f^{(n)}(a)\right| \leq \frac{M_r\, n!}{r^n}\,.$$

Proof. Cauchy's integral formula for the $n$-th derivative gives

$$\left|f^{(n)}(a)\right| = \left|\frac{n!}{2\pi i}\int_{\partial D(a,r)} \frac{f(z)}{(z-a)^{n+1}}\, dz\right| \leq \frac{n!}{2\pi}\int_{\partial D(a,r)} \frac{|f(z)|}{|z-a|^{n+1}}\, |dz| \leq \frac{n!}{2\pi}\int_{\partial D(a,r)} \frac{M_r}{r^{n+1}}\, |dz| = \frac{M_r\, n!}{r^n}\,. \quad ♯$$

Cauchy's inequality has the following consequences:

**Corollary 4 (Liouville's theorem).** *If $f$ is an entire function (i.e., holomorphic on $\mathbf{C}$) which is bounded, then is constant.*

**Corollary 5 (Fundamental Theorem of Algebra).** *Every polynomial (with complex coefficients) of degree $n \geq 1$ has $n$ complex zeros (taking into account the multiplicity of its zeros).*

Recall that a set $K \subseteq \mathbf{C}$ (or $K \subseteq \mathbf{R}^n$) is compact if and only if it is closed and bounded.

**Theorem 7.** *Let $\Omega \subseteq \mathbf{C}$ be an open set and $f_n : \Omega \longrightarrow \mathbf{C}$ a sequence of holomorphic functions which are uniformly convergent on compact sets contained in $\Omega$ to a function $f$. Then $f$ is holomorphic on $\Omega$ and the derivatives $\{f_n^{(k)}\}_n$ uniformly converge on compact sets contained in $\Omega$ to the derivative $f^{(k)}$ for each $k \geq 1$.*

**Application: Laplace transform.** *If $f : [0,\infty) \longrightarrow \mathbf{C}$ is a measurable function with $\lim_{x\to\infty} f(x)\, e^{-ax} = 0$ for some $a \in \mathbf{R}$ and $f \in L^1([0,n])$ for every $n$, then the function (the Laplace transform of $f$)*

$$Lf(z) = \int_0^\infty f(x)\, e^{-zx}\, dx$$

*is holomorphic on the halfplane $\{z \in \mathbf{C} : \; \operatorname{Re} z > a\}$.*

PROOF. For each $R > 0$, $z \in \Omega_R = \{w \in \mathbf{C} : \operatorname{Re} w > -R\}$, $n \in \mathbf{N}$ and $0 \le x \le n$, we have

$$\big|f(x)\, e^{-zx}\big| = \big|f(x)\big| e^{\operatorname{Re}(-zx)} = \big|f(x)\big| e^{x\operatorname{Re}(-z)} \le \big|f(x)\big|\, e^{Rx} \le \big|f(x)\big|\, e^{Rn}.$$

Thus,

$$F_n(z) = \int_0^n f(x)\, e^{-zx}\, dx$$

are holomorphic functions on $\Omega_R$ by the corollary of Morera's theorem. Therefore, $F_n$ are entire functions.

Since $\lim_{x\to\infty} f(x)\, e^{-ax} = 0$, there exists $M$ such that $|f(x)| \le e^{ax}$ for every $x \ge M$. If $n \ge M$, then

$$\big|F_n(z) - Lf(z)\big| = \Big|\int_n^\infty f(x)\, e^{-zx}\, dx\Big| \le \int_n^\infty e^{ax}\, e^{-x\operatorname{Re} z}\, dx = \int_n^\infty e^{(a-\operatorname{Re} z)x} dx\,.$$

Fix $\varepsilon > 0$. If $\operatorname{Re} z \ge a + \varepsilon$, then

$$\big|F_n(z) - Lf(z)\big| \le \int_n^\infty e^{(a-\operatorname{Re} z)x}\, dx \le \int_n^\infty e^{-\varepsilon x}\, dx = \frac{-1}{\varepsilon}\big[e^{-\varepsilon x}\big]_{x=n}^{x=\infty} = \frac{1}{\varepsilon}\, e^{-\varepsilon n} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Note that this gives that $f(x)e^{-zx} \in L^1([0,\infty))$, since $f \in L^1([0,n])$ for every $n$.

Hence, $\{F_n\}$ uniformly converges to $Lf$ on $\{z : \operatorname{Re} z \ge a+\varepsilon\}$, and so, it uniformly converges on compact sets contained in $\{z : \operatorname{Re} z > a\}$. Now the result follows from Theorem 7. ♯

The following interesting result gives that Cauchy's integral theorem with singularities is not a generalization of Cauchy's integral theorem, since the functions on the hypotheses of Cauchy's integral theorem with singularities are, in fact, holomorphic in the whole domain.

**Theorem 8.** *Let $\Omega$ be an open set and $a \in \Omega$. If $f$ is holomorphic on $\Omega \setminus \{a\}$ and $\lim_{z\to a}(z-a)\, f(z) = 0$, then there exists the limit of $f$ as $z$ approaches $a$, and if we define*

$$f(a) := \lim_{z\to a} f(z)\,,$$

*then $f$ is holomorphic on $\Omega$.*

PROOF. Let $D$ be a disk such that $a \in D$ and $\overline{D} \subset \Omega$. The function

$$g(z) := \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\, dw$$

is holomorphic on $D$ (by Theorem 4). Furthermore, Cauchy's integral formula (Theorem 1) can be generalized (with the same proof) to holomorphic functions with singularities as in the statement by using Cauchy's integral theorem with singularities. Hence,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\, dw = g(z)\,, \qquad \text{for every} \quad z \in D \setminus \{a\}\,.$$

Thus, $\lim_{z\to a} f(z) = \lim_{z\to a} g(z) = g(a)$. If we define $f(a) := \lim_{z\to a} f(z) = g(a)$, then $f(z) = g(z)$ for every $z \in D$ and so, $f$ is holomorphic on $\Omega$. ♯

**Definition.** *A point $a$ is a <u>removable singularity</u> of $f$ if $f$ is holomorphic on $\Omega\setminus\{a\}$ and $\lim_{z\to a}(z-a)f(z)=0$.*

EXAMPLES. (1) $z = 0$ is a removable singularity of the function $f(z) = (e^z - 1)/z$. Note that this implies the following: the real function $f$ defined as $f(x) = (e^x - 1)/x$ if $x \ne 0$ and $f(0) = 1$ is of class $C^\infty$ on $\mathbf{R}$.

(2) Prove that $\lim_{z\to 0} z \log z = 0$. Is $z = 0$ a removable singularity of the function $\log z$? It is possible to apply Theorem 8?

We know that every analytic function is holomorphic. The following theorem gives that the converse also holds. Therefore, it answers to the two following questions:

What functions are analytic?

What is the radius of convergence of the series?

**Theorem 9 (Taylor's theorem).** *A function $f$ is holomorphic on the open set $\Omega$ if and only if $f$ is analytic on $\Omega$. Furthermore, if $a \in \Omega$ and $r$ is the minimum distance from $\partial\Omega$ to the point $a$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \qquad \textit{for every } z \in D(a,r)\,.$$

*Also, the radius of convergence at the point $a$ is the distance from $a$ to its closest singularity of $f$.*

PROOF. We know that every analytic function is holomorphic.

Assume now that $f$ is holomorphic on $\Omega$ and fix $a \in \Omega$. Consider $r = \operatorname{dist}(a, \partial\Omega)$ and fix $\rho < \rho' < r$. We are going to prove that $f$ is a power series on the disk $D(a,\rho)$.

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\,\frac{1}{1-\dfrac{z-a}{w-a}} = \frac{1}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\,.$$

If $z \in D(a,\rho)$, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|w-a|=\rho'} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{|w-a|=\rho'} f(w)\left(\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right)dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w-a|=\rho'}\frac{f(w)}{(w-a)^{n+1}}\,dw\right)(z-a)^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n\,, \end{aligned}$$

by Theorem 2. As we can choose any $\rho < r$, then $f$ is equal to its Taylor series for every $z \in D(a,r)$. Therefore, $f$ is analytic at $a$ for every $a \in \Omega$. $\sharp$

This result has several important consequences:

**Corollary 6.** *If $f$ is holomorphic on the domain $\Omega$, and there exists a point $a \in \Omega$ such that*

$$0 = f'(a) = \cdots = f^{(n)}(a) = \cdots\,, \qquad \textit{for every } n \geq 1\,,$$

*then $f$ is constant on $\Omega$.*

PROOF. If $\Omega$ was a disk with center at $a$, the corollary would be a direct consequence of Theorem 9. In the general case, since $\Omega$ is connected, we can join any point $z \in \Omega$ with $a$ by a curve contained in $\Omega$. Since this curve is compact, we can cover it with a finite number of disks centered at points in the curve, and contained in $\Omega$, and such that the center of each disk is contained in the previous disk. By applying Theorem 15, we can prove successively that $f$ is constant ($= f(a)$) on each disk, starting with the disk centered at $a$ and finishing at the disk centered at $z$. Hence, $f(z) = f(a)$ for every $z \in \Omega$. $\sharp$

EXAMPLE?: The Taylor series at $x = 0$ of $f(x) = e^{-1/x^2}$ is 0, but $f$ is not constant.

**Corollary 7.** *If $f$ is holomorphic on the domain $\Omega$ and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ contained in $\Omega$ with $a_n \neq a_m$ if $n \neq m$, and such that*

$$\exists \lim_{n\to\infty} a_n = a \in \Omega\,, \qquad \textit{and} \qquad f(a_n) = 0\,, \qquad n \geq 1\,,$$

*then $f(z) = 0$ for every $z \in \Omega$.*

PROOF. Since $a_n \neq a_m$ if $n \neq m$, we have $a_n = a$ for at most one $n$; hence, we can assume that $a_n \neq a$ for every $n$. Note that $f(a) = 0$, since $f$ is continuous at $a$. If $f^{(n)}(a) = 0$ for every $n$, then Corollary 1 gives that $f \equiv \text{constant} \equiv f(a) = 0$. Otherwise, we can define

$$k = \min\{n :\ f^{(n)}(a) \neq 0\}\,.$$

Thus, by Theorem 9,

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(a)}{n!}\,(z-a)^n = (z-a)^k g(z)\,,$$

where

$$g(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(a)}{n!}\,(z-a)^{n-k} = \sum_{n=0}^{\infty} \frac{f^{(n+k)}(a)}{(n+k)!}\,(z-a)^n\,, \qquad g(a) = \frac{f^{(k)}(a)}{k!} \neq 0\,.$$

In particular, $g$ is holomorphic on some disk $D(a,R)$ (with the same radius of convergence of the series that defines $f$ at $a$), and so, it is continuous at $a$. Since

$$g(a_n) = \frac{f(a_n)}{(a_n-a)^k} = 0\,,$$

we have $g(a) = \lim_{n\to\infty} g(a_n) = 0$, a contradiction. ♯

The following result is a consequence of the argument in the proof of Corollary 7. Since it is very important, it must be stated in an independent way:

**Corollary 8 (Zeros of holomorphic functions).** *If $f$ is holomorphic and non-constant on the open set $\Omega$, and $a \in \Omega$ is a zero of $f$ (i.e., $f(a) = 0$), then there exists a minimum integer $k > 0$ such that*

$$f(z) = (z-a)^k g(z)\,,$$

*where $g$ is holomorphic on $\Omega$, and $g(a) \neq 0$.*

**Corollary 9 (Principle of analytic continuation).** *Let $\Omega \subseteq \mathbf{C}$ be a domain, and $\{a_n\}_{n=1}^{\infty}$ a sequence of different points in $\Omega$ such that*

$$\exists \lim_{n\to\infty} a_n = a \in \Omega\,.$$

*If $f$ and $g$ are holomorphic functions on $\Omega$ such that*

$$f(a_n) = g(a_n)\,, \qquad \text{for every} \;\; n \geq 1\,,$$

*then $f(z) = g(z)$ for every $z \in \Omega$.*

Proof. It suffices to apply the previous corollary to the function $f - g$. ♯

As a direct consequence, we obtain the following result:

**Corollary 10.** *If $\Omega \subseteq \mathbf{C}$ is a domain with $\Omega \cap \mathbf{R} \neq \emptyset$, and $f_1$, $f_2$ are two holomorphic functions on $\Omega$ such that*

$$f_1|_{\Omega\cap\mathbf{R}} = f_2|_{\Omega\cap\mathbf{R}}\,,$$

*then $f_1 = f_2$ on $\Omega$. Consequently, if $f$ is a function defined on an interval contained in a domain $\Omega \subseteq \mathbf{C}$ and $f$ can be analytically extended to $\Omega$, then this extension is unique.*

Therefore, for instance, there is only one way to analytically extend the real exponential function to the complex plane, and we know this extension: $e^z = e^x(\cos y + i \sin y)$.

**Definition.** *Under the hypotheses in the previous corollary, we say that $f$ has a zero of order $k$ at $a$.*

**Corollary 11 (Bernouilli-l'Hôpital's rule).** *Let $f_1$, $f_2$ be two holomorphic functions at $a$, such that $f_1(a) = f_2(a) = 0$. Then*

$$\lim_{z\to a} \frac{f_1(z)}{f_2(z)} = \lim_{z\to a} \frac{f_1'(z)}{f_2'(z)}\,.$$

Proof. Let $k_1$ and $k_2$ be the order of the zero at $a$ of $f_1$ and $f_2$, respectively. Thus,

$$f_1(z) = \sum_{n=k_1}^{\infty} \frac{f_1^{(n)}(a)}{n!}\,(z-a)^n, \quad f_2(z) = \sum_{n=k_2}^{\infty} \frac{f_2^{(n)}(a)}{n!}\,(z-a)^n, \quad f_1^{(k_1)}(a) \neq 0\,,\, f_2^{(k_2)}(a) \neq 0\,.$$

If $k_1 = k_2$, then

$$\lim_{z\to a}\frac{f_1(z)}{f_2(z)} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{n!}\,(z-a)^n}{\sum_{n=k_1}^{\infty}\frac{f_2^{(n)}(a)}{n!}\,(z-a)^n} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{n!}\,(z-a)^{n-k_1}}{\sum_{n=k_1}^{\infty}\frac{f_2^{(n)}(a)}{n!}\,(z-a)^{n-k_1}} = \frac{\frac{f_1^{(k_1)}(a)}{k_1!}}{\frac{f_2^{(k_1)}(a)}{k_1!}} = \frac{f_1^{(k_1)}(a)}{f_2^{(k_1)}(a)}\,,$$

$$\lim_{z\to a}\frac{f_1'(z)}{f_2'(z)} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{(n-1)!}\,(z-a)^{n-1}}{\sum_{n=k_1}^{\infty}\frac{f_2^{(n)}(a)}{(n-1)!}\,(z-a)^{n-1}} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{(n-1)!}\,(z-a)^{n-k_1}}{\sum_{n=k_1}^{\infty}\frac{f_2^{(n)}(a)}{(n-1)!}\,(z-a)^{n-k_1}} = \frac{f_1^{(k_1)}(a)}{f_2^{(k_1)}(a)}\,.$$

If $k_1 > k_2$, then

$$\lim_{z\to a}\frac{f_1(z)}{f_2(z)} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{n!}\,(z-a)^n}{\sum_{n=k_2}^{\infty}\frac{f_2^{(n)}(a)}{n!}\,(z-a)^n} = \lim_{z\to a}(z-a)^{k_1-k_2}\lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{n!}\,(z-a)^{n-k_1}}{\sum_{n=k_2}^{\infty}\frac{f_2^{(n)}(a)}{n!}\,(z-a)^{n-k_2}}$$

$$= 0\cdot\frac{\frac{f_1^{(k_1)}(a)}{k_1!}}{\frac{f_2^{(k_2)}(a)}{k_2!}} = 0\,,$$

$$\lim_{z\to a}\frac{f_1'(z)}{f_2'(z)} = \lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{(n-1)!}\,(z-a)^n}{\sum_{n=k_2}^{\infty}\frac{f_2^{(n)}(a)}{(n-1)!}\,(z-a)^n} = \lim_{z\to a}(z-a)^{k_1-k_2}\lim_{z\to a}\frac{\sum_{n=k_1}^{\infty}\frac{f_1^{(n)}(a)}{(n-1)!}\,(z-a)^{n-k_1}}{\sum_{n=k_2}^{\infty}\frac{f_2^{(n)}(a)}{(n-1)!}\,(z-a)^{n-k_2}} = 0\,.$$

If $k_1 < k_2$, the proof is similar. $\sharp$

**Corollary 12 (Bernouilli-l'Hôpital's rule at infinity).** *Let $f_1$, $f_2$ be two holomorphic functions on $\{|z| > r\}$ for some $r > 0$, such that $\lim_{z\to\infty} f_1(z) = \lim_{z\to\infty} f_2(z) = 0$. Then*

$$\lim_{z\to\infty}\frac{f_1(z)}{f_2(z)} = \lim_{z\to\infty}\frac{f_1'(z)}{f_2'(z)}\,.$$

Proof. The functions $f_1(1/z)$ and $f_2(1/z)$ are holomorphic on a neighborhood of 0 (they have a removable singularity at 0 since $\lim_{z\to 0} f_1(1/z) = \lim_{z\to 0} f_2(1/z) = 0$). Hence, they satisfies the hypotheses on Bernouilli-l'Hôpital's rule at 0 and

$$\lim_{z\to\infty}\frac{f_1(z)}{f_2(z)} = \lim_{w\to 0}\frac{f_1(1/w)}{f_2(1/w)} = \lim_{w\to 0}\frac{-w^{-2}f_1'(1/w)}{-w^{-2}f_2'(1/w)} = \lim_{z\to\infty}\frac{f_1'(z)}{f_2'(z)}\,. \qquad \sharp$$

**Theorem 10 (Open mapping theorem).** *If $f$ is holomorphic on the open set $\Omega$ and it is non-constant, then for each $a \in \Omega$ there exists $\varepsilon > 0$ such that $D(f(a), \varepsilon) \subseteq f(\Omega)$.*

**Theorem 11 (Maximum modulus principle).** *If $f$ is holomorphic on the open set $\Omega$ and it is not constant, then $|f|$ does not have a maximum value on $\Omega$.*

**Corollary 13.** *If $f$ is holomorphic on the bounded open set $\Omega$ and it is continuous on $\overline{\Omega} = \Omega \cup \partial\Omega$, then the maximum value of $|f|$ on $\overline{\Omega}$ is attained on $\partial\Omega$.*