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Departamento de Matemáticas

Complex variable and transforms

Chapter 2: Transforms

Section 2.1: Fourier series

Professors:

Domingo Pestana Galván

José Manuel Rodríguez García

1 Fourier series

1.1 Square integrable functions

We define the space of square integrable functions on an interval $[a,b]$, denoted by $L^2[a,b]$, as

$$L^2[a,b] = \Big\{ f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 \, dx < \infty \Big\}.$$

If we identify functions that are equal almost everywhere ($f = g$ a.e., that is, the set $\{f \neq g\}$ has zero length), then

$$\|f\|_2 = \Big(\int_a^b |f(x)|^2 \, dx \Big)^{1/2}$$

defines a norm on $L^2[a,b]$, i.e.,

- $\|f\|_2 \geq 0$ for every function $f \in L^2[a,b]$, and $\|f\|_2 = 0 \iff f = 0$.
- $\|\alpha f\|_2 = |\alpha| \, \|f\|_2$ for every function $f \in L^2[a,b]$, and every scalar α.
- The triangle inequality holds: $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$ for every functions $f, g \in L^2[a,b]$.

Recall that a norm induces a distance, called its (norm) induced metric, by the formula $d(f,g) = \|f - g\|_2$, which make the normed vector space $L^2[a,b]$ into a metric space:

- $d(f,g) \geq 0$ for every functions $f, g \in L^2[a,b]$, and $d(f,g) = 0 \iff f = g$.
- The triangle inequality holds: $d(f,h) \leq d(f,g) + d(g,h)$ for every functions $f, g, h \in L^2[a,b]$.

Also, $L^2[a,b]$ is an inner product space with inner product given by

$$\langle f, g \rangle = \int_a^b f(x) \, \overline{g(x)} \, dx,$$

that is,

- $\langle f, f \rangle \geq 0$ for every function $f \in L^2[a,b]$, and $\langle f, f \rangle = 0 \iff f = 0$.
- $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for every functions $f, g, h \in L^2[a,b]$, and every scalars α, β.
- $\langle g, f \rangle = \overline{\langle f, g \rangle}$ for every functions $f, g \in L^2[a,b]$.

We say that $\|\cdot\|_2$ is the induced norm by the inner product $\langle \cdot, \cdot \rangle$, that is, $\|f\|_2 = \langle f, f \rangle$ for every $f \in L^2[a,b]$.

The following important properties hold:

- $\|f + g\|_2 \geq \big| \|f\|_2 - \|g\|_2 \big|$ for every functions $f, g \in L^2[a,b]$.
- $\big\| \sum_{k=1}^n f_k \big\|_2 \leq \sum_{k=1}^n \|f_k\|_2$ for every functions $f_1, \ldots, f_n \in L^2[a,b]$.
- $f / \|f\|_2$ has norm 1 for every function $f \in L^2[a,b] \setminus \{0\}$.
- $\big| \langle f, g \rangle \big| \leq \|f\|_2 \|g\|_2$ for every functions $f, g \in L^2[a,b]$ (Cauchy-Schwarz inequality).
- $\|f + g\|_2^2 + \|f - g\|_2^2 = 2\|f\|_2^2 + 2\|g\|_2^2$ for every functions $f, g \in L^2[a,b]$ (parallelogram law).
- $\langle f, g \rangle = \frac{1}{4} \big(\|f + g\|_2^2 - \|f - g\|_2^2 + i\|f + ig\|_2^2 - i\|f - ig\|_2^2 \big)$ for every functions $f, g \in L^2[a,b]$ (polarization identity).
- $\langle f, g \rangle = \frac{1}{4} \big(\|f + g\|_2^2 - \|f - g\|_2^2 \big)$ for every functions $f, g \in L^2[a,b]$ with real values.

It can be shown that square integrable functions form a complete metric space under the metric induced by the inner product defined above, that is, the sequences in such metric spaces converge if and only if they are Cauchy sequences. $L^2[a,b]$ is a Hilbert space, that is, it is a complete space under the metric induced by the inner product.

We say that $\{e_n\}_{n=1}^{\infty}$ is an orthogonal set if $\langle e_n, e_m\rangle = 0$ for every $n \neq m$; $\{e_n\}_{n=1}^{\infty}$ is an orthonormal set if it is orthogonal and $\|e_n\|^2 = \langle e_n, e_n\rangle = 1$ for every n.

We say that $\{e_n\}_{n=1}^{\infty}$ is an orthogonal basis of $L^2[a,b]$ if it is an orthogonal set and every element f of $L^2[a,b]$ may be written as

$$f = \sum_{n=1}^{\infty} \frac{\langle f, e_n\rangle}{\|e_n\|^2}\, e_n .$$

When $\{e_n\}_{n=1}^{\infty}$ is orthonormal, we have

$$f = \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n .$$

This sum is called the Fourier expansion of f, and has the following consequences

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \frac{\langle f, e_n\rangle \overline{\langle g, e_n\rangle}}{\|e_n\|^2}\,,$$

$$\int_a^b f(x)\,\overline{g(x)}\,dx = \sum_{n=1}^{\infty} \frac{\int_a^b f(x)\,\overline{e_n(x)}\,dx\;\overline{\int_a^b g(x)\,\overline{e_n(x)}\,dx}}{\int_a^b |e_n(x)|^2\,dx}\,.$$

and

$$\langle f, f\rangle = \sum_{n=1}^{\infty} \frac{\big|\langle f, e_n\rangle\big|^2}{\|e_n\|^2}\,,$$

$$\int_a^b |f(x)|^2\,dx = \sum_{n=1}^{\infty} \frac{\big|\int_a^b f(x)\,\overline{e_n(x)}\,dx\big|^2}{\int_a^b |e_n(x)|^2\,dx}\,.$$

which is called Parseval's identity.

1.2 Definitions and previous results

Given a function $f : [-L, L] \longrightarrow \mathbb{R}$, its Fourier series is defined as:

$$\frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,,$$

where their Fourier coefficients are defined by the formulas:

$$a_n = a_n(f) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\,, \quad n \geq 0\,,$$

$$b_n = b_n(f) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, \quad n \geq 1\,.$$

Note that these coefficients are well defined if f is integrable on the interval $[-L, L]$.

The first problem we have to ask ourselves is under what conditions the Fourier series converge. In order to obtain the simplest convergence criterion for such series we need to introduce some definitions.

We say that $f(x)$ has a jump discontinuity at the point $x = a$, if there exist both the left-hand limit $\lim_{x\to a^-} f(x) = f(a-)$ as well as the right-hand limit $\lim_{x\to a^+} f(x) = f(a+)$, and they are distinct.

A function $f(x)$ is said to be piecewise continuous on $[-L, L]$ if the interval can be divided into subintervals, such that $f(x)$ is continuous on each open subinterval, and the one-sided limits exist at the ends of the subintervals. Therefore, at the ends of the subintervals $f(x)$ can either be continuous or have jump discontinuities.

A function $f(x)$ is said to be piecewise continuously differentiable or piecewise C^1 on $[-L, L]$ if $f(x)$ and $f'(x)$ are piecewise continuous on that interval.

A function $f(x)$ is said to be periodic with period P or P-periodic if $f(x + P) = f(x)$ for all x in the domain of f.

The Fourier series of $f(x)$ on $[-L, L]$ is a $2L$-periodic function, but since $f(x)$ is not necessarily periodic, to study the convergence of the series it is necessary to consider the *periodic extension* of $f(x)$.

Since a function $f(x)$ can be different from its Fourier series on the interval $[-L, L]$, since the series may not converge, and if it does converge, it may not converge to $f(x)$, instead of the equality symbol we will use the notation

$$f(x) \sim \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\ ,$$

where the symbol $\sim$ means that the series is the Fourier series of $f(x)$, even if the series diverges or converges at some point x to a value other than $f(x)$.

Theorem 1.1 *(Convergence theorem for Fourier series) If $f(x)$ is piecewise continuously differentiable on $[-L, L]$, then the Fourier series of $f(x)$ converges:*
• to the periodic extension of f, at those points x for which the periodic extension of f is continuous,
• to the average of the two side limits of the periodic extension of f

$$\frac{1}{2}\big(f(x+) + f(x-)\big),$$

at those points x for which the periodic extension of f has a jump discontinuity (here, the periodic extension of f is denoted by f also).

Next we will consider the series that only contain sines and those that only contain cosines, which are special cases of Fourier series.

Recall that a function $f(x)$ is odd if $f(-x) = -f(x)$, and $f(x)$ is even if $f(-x) = f(x)$.

The Fourier coefficients of an odd function verify that $a_n = 0$ for all $n \geq 0$, since the integrand $f(x)\cos(n\pi x/L)$ that appears in the definition of these coefficients is an odd function. Therefore, the cosine functions do not appear in the Fourier series of an odd function. The Fourier series of an odd function is a sine series (which is an odd function):

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\ ,$$

where the coefficients verify:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, \quad n \geq 1\,.$$

When a function f is defined on $[0, L]$ and we need its Fourier sine series we define the <u>periodic odd extension</u> of f as the function of period $2L$ such that

$$F(x) = \begin{cases} f(x), & x \in [0, L], \\ -f(-x), & x \in (-L, 0)\,, \end{cases}$$

and the Fourier series on $[-L, L]$ of this odd extension is the Fourier sine series of f on $[0, L]$.

Example 1.2 *The odd extension of $f(x) = x$ on $[0, L]$ to $[-L, L]$ is $F(x) = x$ and its Fourier sine series on $[0, L]$ is*

$$x \sim \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{L}.$$

If f is an even function, by the symmetry, $b_n = 0$ for every $n \geq 1$, so we get that the Fourier series of an even function is a cosine series

$$f(x) \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\,,$$

with coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx.$$

Given a function f on $[0, L]$, when we need its Fourier cosine series we define the <u>periodic even extension</u> of f as the function of period $2L$ such that

$$F(x) = \begin{cases} f(x), & x \in [0, L], \\ f(-x), & x \in (-L, 0)\,, \end{cases}$$

and the Fourier series of this extension to $[-L, L]$ is the Fourier cosine series of f on $[0, L]$.

Example 1.3 *The even extension of $f(x) = x$ on $[0, L]$ to $[-L, L]$ is $F(x) = |x|$ and its Fourier cosine series on $[0, L]$ is*

$$x \sim \frac{L}{2} + \sum_{k=0}^{\infty} \frac{-4L}{(2k+1)^2\pi^2}\cos\frac{(2k+1)\pi x}{L}.$$

As a consequence of the previous definitions and the convergence of the Fourier series we have the following result.

Theorem 1.4 *1. **General Fourier series**. If f is piecewise C^1 on $(-L, L)$, then its Fourier series is continuous if and only if f is continuous on $[-L, L]$ and $f(-L) = f(L)$.*

*2. **Cosine series**. If f is piecewise C^1 on $(0, L)$, then its Fourier cosine series is continuous if and only if f is continuous on $[0, L]$.*

3. ***Sine series***. *If f is piecewise C^1 on $(0, L)$, then its Fourier sine series is continuous if and only if f is continuous on $[0, L]$ and $f(0) = f(L) = 0$.*

Observe that the series that needs more conditions is the sine series and the series that requires less conditions is the cosine series.
When we use Fourier series to solve PDEs it is necessary to derive and to integrate this kind of expressions. Now we study the conditions with which this can be done.

Theorem 1.5

1. ***General Fourier series***. *If the Fourier series of f is continuous and f' is piecewise C^1, then it can be derived term by term, and the series we obtain is the Fourier series of f' (that converges to f' at the points of continuity of f').*

2. ***Fourier cosine series***. *If f is continuous on $[0, L]$ and f' is piecewise C^1 on $(0, L)$, then its Fourier cosine series can be derived term by term, and the series we obtain is the Fourier sine series of f' (that converges to f' at the points of continuity of f').*

3. ***Fourier sine series***. *If f is continuous on $[0, L]$ and f' is piecewise C^1 on $(0, L)$, then its Fourier sine series can be derived term by term if and only if $f(0) = f(L) = 0$. In case we can derive term by term, the series we obtain is the Fourier cosine series of f' (that converges to f' at the points of continuity of f').*

Observe that, again, the sine series are more demanding. Nevertheless, if we have a sine series of a function f that is continuous on $[0, L]$ and piecewise C^1 on $(0, L)$,

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} ,$$

it can be proved that the Fourier cosine series of the derivative is

$$f'(x) \sim \frac{1}{L}\big(f(L) - f(0)\big) + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} b_n + \frac{2}{L}\big((-1)^n f(L) - f(0)\big) \right) \cos \frac{n\pi x}{L} ,$$

and, as can be seen directly, this coincides with the derivative term by term of the Fourier sine series of f if and only if $f(0) = f(L) = 0$.

Theorem 1.6 *If $f : \mathbb{R} \to \mathbb{R}$ is a $2L$-periodic continuous function and f' is piecewise continuous on $[-L, L]$, then the Fourier series converges uniformly to f.*

Definition 1.7 *If $S_N(x; f)$ denotes the N-th Fourier sum of f:*

$$S_N(x; f) = \frac{1}{2} a_0 + \sum_{n=1}^{N} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{N} b_n \sin \frac{n\pi x}{L} ,$$

the N-th Cesàro sum $C_N(x; f)$ of f is defined as:

$$C_N(x; f) = \frac{1}{N+1} \Big(S_0(x; f) + S_1(x; f) + \cdots + S_N(x; f) \Big) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(x; f).$$

Theorem 1.8 *If $f : \mathbb{R} \to \mathbb{R}$ is a $2L$-periodic continuous function, then the sequence of Cesàro partial sums $C_N(x; f)$ converges uniformly to f.*

The above result means that every continuous and $2L$-periodic function in the real line can be uniformly approximated by trigonometric polynomials.

Sometimes we need the derivative with respect to variables that do not appear in the eigenfunctions, and this is easy.

Theorem 1.9 *If $u = u(x,t)$ is a continuous function on $[-L, L] \times [0, \infty)$ and $\partial u/\partial t$ is piecewise C^1 as a function of $x \in (-L, L)$ for every $t \in [0, \infty)$, then its Fourier series*

$$u(x,t) = \frac{1}{2}\, a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}\,,$$

can be derived term by term with respect to the parameter t, and we obtain

$$\frac{\partial u}{\partial t}(x,t) \sim \frac{1}{2}\, a_0'(t) + \sum_{n=1}^{\infty} a_n'(t) \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{L}\,.$$

With respect to integration we have that the three Fourier series can be integrated term by term, and the result is a series that is convergent for all $x \in [-L, L]$ to the integral of f. However, it may happen that the series we obtain is **not** a Fourier series. In fact, given the series

$$f(x) \sim \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}\,,$$

term by term integration gives

$$\int_{-L}^{x} f(t)\, dt \sim \frac{1}{2}\, a_0(x+L) + \sum_{n=1}^{\infty} \left(\frac{a_n L}{n\pi} \sin \frac{n\pi x}{L} + \frac{b_n L}{n\pi} \left((-1)^n - \cos \frac{n\pi x}{L} \right) \right),$$

that, as can be seen easily, is a Fourier series if and only if $a_0 = 0$.

Fourier series have interesting properties; some of them appear in the following formulas (next, we assume that f and g are integrable on $[-L, L]$ and $\alpha, \beta \in \mathbb{R}$):

(FSa) $a_n\big(\alpha f + \beta g\big) = \alpha\, a_n\big(f\big) + \beta\, a_n\big(g\big)$ $(n \geq 0)$,
$b_n\big(\alpha f + \beta g\big) = \alpha\, b_n\big(f\big) + \beta\, b_n\big(g\big)$ $(n \geq 1)$,

(FSb) if f is continuous and f' is piecewise continuous on $[-L, L]$, and $f(-L) = f(L)$, then

$$a_0\big(f'\big) = 0\,, \quad a_n\big(f'\big) = \frac{n\pi}{L}\, b_n\big(f\big) \text{ and } b_n\big(f'\big) = \frac{-n\pi}{L}\, a_n\big(f\big) \ (n \geq 1)\,,$$

(FSc) if f and f' are continuous and f'' is piecewise continuous on $[-L, L]$, $f(-L) = f(L)$, and $f'(-L) = f'(L)$, then

$$a_0\big(f''\big) = 0\,, \quad a_n\big(f''\big) = -\Big(\frac{n\pi}{L}\Big)^2 a_n\big(f\big) \text{ and } b_n\big(f''\big) = -\Big(\frac{n\pi}{L}\Big)^2 b_n\big(f\big) \ (n \geq 1)\,,$$

(FSd) $\displaystyle \sup_{n\geq 0} |a_n| \leq \frac{1}{L} \int_{-L}^{L} \big|f(x)\big|\, dx\,, \quad \sup_{n\geq 1} \big|b_n\big| \leq \frac{1}{L} \int_{-L}^{L} \big|f(x)\big|\, dx\,,$

(FSe) $\displaystyle \frac{1}{L} \int_{-L}^{L} \big|f(x)\big|^2 dx = \frac{1}{2}\, |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 + \sum_{n=1}^{\infty} \big|b_n\big|^2$ (Parseval's identity),

(FSf) $\displaystyle \frac{1}{L} \int_{-L}^{L} f(x)\, g(x)\, dx = \frac{1}{2}\, a_0\big(f\big) a_0\big(g\big) + \sum_{n=1}^{\infty} a_n\big(f\big) a_n\big(g\big) + \sum_{n=1}^{\infty} b_n\big(f\big) b_n\big(g\big)\,.$

TABLE OF FOURIER SERIES $(k \geq 1, a \in \mathbb{R}, R > r > 0)$

$$(FS1) \quad f(x) = 1\,, \quad a_0 = 2\,, \quad a_n = b_n = 0 \ (n \geq 1)\,,$$

$$(FS2) \quad f(x) = \cos\frac{k\pi x}{L}\,, \quad a_k\big(f\big) = 1\,, \quad a_n = 0 \ (n \neq k)\,, \quad b_n = 0 \ (n \geq 1)\,,$$

$$(FS3) \quad f(x) = \sin\frac{k\pi x}{L}\,, \quad a_n = 0 \ (n \geq 0)\,, \quad b_k = 1\,, \quad b_n = 0 \ (n \neq k)\,,$$

$$(FS4) \quad f(x) = x\,, \quad a_n = 0 \ (n \geq 0)\,, \quad b_n = (-1)^{n+1}\frac{2L}{n\pi} \ (n \geq 1)\,,$$

$$(FS5) \quad f(x) = x^2, \quad a_0 = \frac{2}{3}L^2, \quad a_n = (-1)^n\frac{4L^2}{n^2\pi^2} \ (n \geq 1)\,, \quad b_n = 0 \ (n \geq 1)\,,$$

$$(FS6) \quad f(x) = e^{ax}, \quad a_n = (-1)^n\frac{2aL\sinh aL}{a^2L^2 + n^2\pi^2}\,, \quad b_n = (-1)^{n+1}\frac{2\pi n\sinh aL}{a^2L^2 + n^2\pi^2}\,.$$

$$(FS7) \quad f(x) = \frac{R^2 - r^2}{R^2 - 2Rr\cos x + r^2}\,, \ (\text{with } L = \pi)\,, \quad a_n = 2\,\frac{r^n}{R^n}\,, \quad b_n = 0\,.$$

Useful integral formulas for Fourier series $(a, b, k \in \mathbb{R})$

$$(I1) \quad \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0\,, \qquad m \geq 1\,,\ n \geq 0\,,$$

$$(I2) \quad \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0\,, & \text{if } m \neq n\,,\ m, n \geq 0\,, \\ L\,, & \text{if } m = n \geq 1\,, \\ 2L\,, & \text{if } m = n = 0\,, \end{cases}$$

$$(I3) \quad \int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0\,, & \text{if } m \neq n\,,\ m, n \geq 1\,, \\ L\,, & \text{if } m = n \geq 1\,. \end{cases}$$

$$(I4) \quad \int x^k\cos bx\,dx = \frac{1}{b}x^k\sin bx - \frac{k}{b}\int x^{k-1}\sin bx\,dx\,,$$

$$(I5) \quad \int x^k\sin bx\,dx = \frac{-1}{b}x^k\cos bx + \frac{k}{b}\int x^{k-1}\cos bx\,dx\,,$$

$$(I6) \quad \int e^{ax}\cos bx\,dx = \frac{1}{a^2 + b^2}\,e^{ax}\big(b\sin bx + a\cos bx\big)\,,$$

$$(I7) \quad \int e^{ax}\sin bx\,dx = \frac{1}{a^2 + b^2}\,e^{ax}\big(a\sin bx - b\cos bx\big)\,.$$

1.3 Complex Fourier series

Given a function $f : [-L, L] \longrightarrow \mathbb{C}$, its complex Fourier series is defined as:

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}\,,$$

where their complex Fourier coefficients are defined by the formula:

$$c_n = c_n\big(f\big) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\,dx\,, \quad n \in \mathbb{Z}\,.$$

This series can be obtained from the (real) Fourier series, by using the identities

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.
We have in this case the Parseval's identity

$$\int_{-L}^{L} \left|f(x)\right|^2 dx = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 \,,$$

$$\int_{-L}^{L} f(x)\, g(x)\, dx = 2L \sum_{n=-\infty}^{\infty} c_n(f)\overline{c_n(g)}\,.$$

1.4 Application to partial differential equations

A partial differential equation is an equation in which the unknown is a function of several variables and such that the equation relates this function to some of its partial derivatives. In particular, we are going to study partial differential equations of the type $L[u](x,t) = 0$, where L is a linear operator with constant coefficients of the form:

$$L[u] = A\,\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial t^2} + C\,\frac{\partial^2 u}{\partial x \partial t} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial t} + F\,u\,,$$

where $A, B, C, D, E, F \in \mathbb{R}$ and $u = u(x,t)$ is the unknown function we want to find.
We say that L is a linear operator because $L[\alpha u + \beta v] = \alpha L[u] + \beta L[v]$ for every functions u, v and constants $\alpha, \beta \in \mathbb{R}$.
We are going to apply the method of separation of variables, in order to solve the heat equation on a rod with zero temperature at finite ends.
The problem is described by

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0, & \\ u(0,t) = u(L,t) = 0, & t > 0, & \text{(Dirichlet BC)} \\ u(x,0) = f(x), & 0 < x < L. & \text{(IC)} \end{cases}$$

We look for product solutions of the form $u(x,t) = \varphi(x)T(t)$ that satisfy the equation and obtain

$$\frac{T'}{kT} = \frac{\varphi''}{\varphi} = -\lambda,$$

where λ is an arbitrary constant. We also separate the boundary conditions

$$\begin{cases} u(0,t) = \varphi(0)T(t) = 0 \quad \forall t & \Longrightarrow \quad \varphi(0) = 0, \\ u(L,t) = \varphi(L)T(t) = 0 \quad \forall t & \Longrightarrow \quad \varphi(L) = 0, \end{cases}$$

because $T(t) \equiv 0$ implies $u(x,t) \equiv 0$, that is not a valid solution since $u(x,0) = f(x)$. Then, we have:
Time equation. $T' = -\lambda k T \implies T(t) = \mathrm{e}^{-k\lambda t}$.
Eigenvalue problem.

$$\begin{cases} \varphi'' + \lambda\varphi = 0, & 0 < x < L, \\ \varphi(0) = 0,\ \varphi(L) = 0. & \end{cases}$$

We are going to see that for some values of λ there exists a non trivial solution. Since the characteristic equation is $r^2 + \lambda = 0$ there are three different cases:

Case 1. $\lambda < 0 \implies r = \pm\sqrt{-\lambda}$, two different real roots, so for c_1, c_2 arbitrary constants,

$$\begin{aligned}
&\varphi(x) = c_1 \mathrm{e}^{\sqrt{-\lambda}\,x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}\,x},\\
&\varphi(0) = \varphi(L) = 0 \quad \implies \quad c_1 = c_2 = 0 \quad \implies \quad \varphi(x) \equiv 0.
\end{aligned}$$

Case 2. $\lambda = 0 \implies r = 0$, double root,

$$\begin{aligned}
&\varphi(x) = c_1 + c_2 x,\\
&\varphi(0) = \varphi(L) = 0 \quad \implies \quad c_1 = c_2 = 0 \quad \implies \quad \varphi(x) \equiv 0.
\end{aligned}$$

Case 3. $\lambda > 0 \implies r = \pm i\sqrt{\lambda}$, pure imaginary roots,

$$\varphi(x) = c_1 \sin\left(\sqrt{\lambda}\,x\right) + c_2 \cos\left(\sqrt{\lambda}\,x\right)$$

$$\begin{cases}
\varphi(0) = 0 \implies c_2 = 0,\\
\varphi(L) = 0 \implies c_1 \sin\left(\sqrt{\lambda}\,L\right) = 0 \implies \begin{cases} c_1 = 0 \implies \varphi(x) \equiv 0,\\ \sin\left(\sqrt{\lambda}\,L\right) = 0.\end{cases}
\end{cases}$$

The last condition implies that $\sqrt{\lambda}\,L = n\pi,\ n = 1, 2, \ldots$ This gives us the eigenvalues and eigenfunctions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \varphi_n(x) = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, \ldots$$

We have found the product solutions

$$u_n(x,t) = \sin\frac{n\pi x}{L}\,\mathrm{e}^{-k(n\pi/L)^2 t}, \qquad n = 1, 2, \ldots$$

Using the superposition principle, any linear combination will also be a solution. In fact, if the series

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,\mathrm{e}^{-k(n\pi/L)^2 t}$$

converges "properly" then it will be also a solution of the heat equation with zero boundary conditions.
The initial condition is satisfied if we can find the coefficients b_n such that the initial condition can be written in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\,.$$

This expression is a **Fourier sine series** and we have that

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx.$$

Finally, the solution of our heat problem is

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}\,dy\right) e^{-k(n\pi/L)^2 t}\sin\frac{n\pi x}{L}\\
&= \int_0^L f(y)\left(\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi y}{L} e^{-k(n\pi/L)^2 t}\right)\,dy\\
&= \int_0^L f(y)\,G(x,y,t)\,dy.
\end{aligned}$$

The function G is known as **Green function** of the problem.
This method can also be applied with different boundary conditions. It can be applied in the wave equation and in Laplace's equation, even with more variables. The important question is to solve the eigenvalue problem and characterize the orthogonality relations.

1.5 Application to periodic signals: energy of a signal

A signal is a real (or complex) valued function, e.g., voltage across a resistor or current through inductor, pressure at a point in the ocean, etc.
If a signal is $2L$-periodic, we define its mean energy as

$$\frac{1}{2L}\int_{-L}^{L}\left|f(x)\right|^2 dx.$$

By Parseval's identity, we can write the mean energy of a signal in terms of the coefficients of its Fourier series

$$\frac{1}{2L}\int_{-L}^{L}\left|f(x)\right|^2 dx = \frac{1}{4}\left|a_0\right|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left|a_n\right|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left|b_n\right|^2.$$