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Universidad **Carlos III** de Madrid

Departamento de Matemáticas

Complex variable and transforms

Chapter 2: Transforms

Section 2.2: Fourier transform

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2 Fourier transform

We define the space of integrable functions on the real line, denoted by $L^1(\mathbb{R})$, as

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : \|f\|_1 = \int_{-\infty}^{\infty} |f(x)| dx < \infty \right\}.$$

Given $f \in L^1(\mathbb{R})$, we define its Fourier transform as

$$\hat{f}(\omega) = \mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx.$$

In this case, as $|\hat{f}(\omega)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)| dx = \frac{1}{2\pi} \|f\|_1 < \infty$, it is clear that \hat{f} is a bounded function. If $f \notin L^1(\mathbb{R})$ but there exists the principal value of the integral, then \hat{f} is defined as this principal value:

$$\hat{f}(\omega) = \mathcal{F}[f](\omega) = \text{p.v.} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx := \lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R f(x) e^{i\omega x} dx.$$

Definition 2.1 Given $f, g \in L^1(\mathbb{R})$, we define its convolution as

$$(f * g)(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x-y) g(y) dy.$$

It is easy to check that $\|f * g\|_1 \leq \frac{1}{2\pi} \|f\|_1 \|g\|_1$ and so $f * g \in L^1(\mathbb{R})$.

Theorem 2.2 (Properties of Fourier transform). Let $f, g \in L^1(\mathbb{R})$, $\alpha \in \mathbb{R}$, $a, b \in \mathbb{C}$, $I \subseteq \mathbb{R}$ an open interval. Then

- (1) $\mathcal{F}[af + bg](\omega) = a \mathcal{F}[f](\omega) + b \mathcal{F}[g](\omega)$.
- (2) $\mathcal{F}[e^{i\alpha x} f(x)](\omega) = \mathcal{F}[f](\omega + \alpha)$.
- (3) $\mathcal{F}[f(x - \alpha)](\omega) = e^{i\alpha\omega} \mathcal{F}[f](\omega)$.
- (4) $\mathcal{F}[f(\alpha x)](\omega) = \frac{1}{|\alpha|} \mathcal{F}[f]\left(\frac{\omega}{\alpha}\right)$.
- (5) $\mathcal{F}[\bar{f}](\omega) = \overline{\mathcal{F}[f](-\omega)}$.
- (6) $\mathcal{F}[f * g](\omega) = \mathcal{F}[f](\omega) \mathcal{F}[g](\omega)$.
- (7) If $xf(x) \in L^1(\mathbb{R})$, then \hat{f} is derivable and

$$\mathcal{F}[xf(x)](\omega) = -i \frac{d}{d\omega} (\mathcal{F}[f](\omega)).$$

- (8) If f is derivable, f' is continuous and $f' \in L^1(\mathbb{R})$, then

$$\mathcal{F}[f'](\omega) = -i\omega \mathcal{F}[f](\omega).$$

- (9) If $f(\cdot, t) \in L^1(\mathbb{R})$ for all $t \in I$, $\exists \frac{\partial f}{\partial t}(x, t)$ a.e. $x \in \mathbb{R}$ and for all $t \in I$, and $\exists F \in L^1(\mathbb{R})$ such that $|\frac{\partial f}{\partial t}(x, t)| \leq F(x)$ a.e. $x \in \mathbb{R}$ and for all $t \in I$, then

$$\mathcal{F}\left[\frac{\partial f}{\partial t}(x, t)\right](\omega) = \frac{\partial}{\partial t} (\mathcal{F}[f](\omega)).$$

Corollary 2.3 If $f \in L^1(\mathbb{R})$ takes real values, then $\mathcal{F}[f](\omega) = \overline{\mathcal{F}[f](-\omega)}$.

A direct consequence of (8) is the following

Corollary 2.4 If f is derivable, f' is continuous and $f, f' \in L^1(\mathbb{R})$, then $\lim_{|\omega| \rightarrow \infty} \hat{f}(\omega) = 0$.

This is a weak version of the following result:

Lemma 2.5 (Riemann-Lebesgue lemma). If $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$, i.e. \hat{f} is continuous and vanishes at infinity:

$$\lim_{|\omega| \rightarrow \infty} \hat{f}(\omega) = 0.$$

The following result shows that the Fourier transform can be inverted and that its inverse is essentially a Fourier transform:

Theorem 2.6 (Inversion of Fourier transform). If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) := \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\omega x} d\omega.$$

In particular, $f \in C_0(\mathbb{R})$.

2.1 Fourier transform on $L^2(\mathbb{R})$

We define the space of square integrable functions on the real line, denoted by $L^2(\mathbb{R})$, as

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : \|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2} < \infty \right\}.$$

As $C_c(\mathbb{R}) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $C_c(\mathbb{R})$ is dense on $L^2(\mathbb{R})$ we have that also $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense on $L^2(\mathbb{R})$ and so, given $f \in L^2(\mathbb{R})$,

$$\exists \{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \text{ such that } \|f_n - f\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, the definition of $\hat{f}(\omega)$ applies for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. It turns out that in this case $\hat{f} \in L^2(\mathbb{R})$ and, in fact, Plancherel's theorem holds:

$$\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}} \|f\|_2, \quad \forall f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$

i.e. Fourier transform preserves L^2 -norm in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ up to a multiplicative constant. This fact allows us to extend Fourier transform to the whole $L^2(\mathbb{R})$: Given $f \in L^2(\mathbb{R})$ let $\{f_n\}_{n=1}^{\infty} \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be a sequence such that $\|f_n - f\|_2 \rightarrow 0$ as $n \rightarrow \infty$. But then $\{\hat{f}_n\}_{n=1}^{\infty}$ is a Cauchy sequence and as

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|\mathcal{F}(f_n - f_m)\|_2 = \frac{1}{\sqrt{2\pi}} \|f_n - f_m\|_2$$

and so we also have that $\{\hat{f}_n\}_{n=1}^{\infty}$ is also a Cauchy sequence and therefore converges to a function $g \in L^2(\mathbb{R})$ since $L^2(\mathbb{R})$ is complete. We define $\hat{f} = g$. But, as $\|\hat{f}_n - g\|_2 \rightarrow 0$ as $n \rightarrow \infty$, we also have

$$\|f_n\|_2 \rightarrow \|f\|_2, \quad \|\hat{f}_n\|_2 \rightarrow \|g\|_2, \quad \|\hat{f}_n\|_2 = \frac{1}{\sqrt{2\pi}} \|f_n\|_2.$$

Hence, as the limit of a sequence is unique, we obtain Plancherel's theorem for all functions in $L^2(\mathbb{R})$:

$$\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}} \|f\|_2, \quad \forall f \in L^2(\mathbb{R}).$$

Theorem 2.7 One can associate to each $f \in L^2(\mathbb{R})$ a function $\hat{f} \in L^2(\mathbb{R})$ so that the following properties hold:

- (1) If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then \hat{f} is the previously defined Fourier transform.
- (2) For every $f \in L^2(\mathbb{R})$, $\|\hat{f}\|_2 = \frac{1}{\sqrt{2\pi}} \|f\|_2$, i.e.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega. \quad (\text{Plancherel's theorem})$$

- (3) The mapping $f \mapsto \hat{f}$ is a Hilbert's space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$: $\frac{1}{2\pi} \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$, i.e.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx = \int_{-\infty}^{\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} d\omega, \quad \forall f, g \in L^2(\mathbb{R}).$$

- (4) The following symmetric relation exists between f and \hat{f} . If:

$$\varphi_R(\omega) = \frac{1}{2\pi} \int_{-R}^R f(x) e^{i\omega x} dx, \quad \psi_R(\omega) = \int_{-R}^R \hat{f}(\omega) e^{-i\omega x} d\omega,$$

then

$$\|\varphi_R - \hat{f}\|_2 \rightarrow 0 \quad \text{and} \quad \|\psi_R - f\|_2 \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

FOURIER TRANSFORMS TABLE

$(x_0 \in \mathbb{R}, \alpha, \beta > 0)$

$$(TF1) \quad \mathcal{F}[e^{-\alpha x^2}](\omega) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/(4\alpha)},$$

$$(TF2) \quad \mathcal{F}\left[\sqrt{\frac{\pi}{\alpha}} e^{-x^2/(4\alpha)}\right](\omega) = e^{-\alpha\omega^2},$$

$$(TF3) \quad \mathcal{F}[e^{-\alpha|x|}](\omega) = \frac{\alpha}{\pi(\omega^2 + \alpha^2)},$$

$$(TF4) \quad \mathcal{F}\left[\frac{2\alpha}{x^2 + \alpha^2}\right](\omega) = e^{-\alpha|\omega|},$$

$$(TF5) \quad \mathcal{F}[\chi_{[-\alpha, \alpha]}(x)](\omega) = \frac{\sin \alpha\omega}{\pi\omega},$$

$$(TF6) \quad \mathcal{F}\left[\frac{\sin \alpha x}{x}\right](\omega) = \frac{1}{2} \chi_{[-\alpha, \alpha]}(\omega),$$

$$(TF7) \quad \mathcal{F}[x\chi_{[-\alpha, \alpha]}(x)](\omega) = i \frac{\sin \alpha\omega - \alpha\omega \cos \alpha\omega}{\pi\omega^2},$$

$$(TF8) \quad \mathcal{F}[\chi_{[0, \alpha]}(x) - \chi_{[-\alpha, 0]}(x)](\omega) = i \frac{1 - \cos \alpha\omega}{\pi\omega},$$

$$(TF9) \quad \mathcal{F}[|x|\chi_{[-\alpha, \alpha]}(x)](\omega) = \frac{\alpha\omega \sin \alpha\omega + \cos \alpha\omega - 1}{\pi\omega^2},$$

$$(TF10) \quad \mathcal{F}[(\alpha - |x|)\chi_{[-\alpha, \alpha]}(x)](\omega) = \frac{1 - \cos \alpha\omega}{\pi\omega^2} = \frac{\sin^2(\alpha\omega/2)}{2\pi\omega^2},$$

$$(TF11) \quad \mathcal{F}[e^{-i\alpha x^2}](\omega) = \frac{1}{\sqrt{4\pi\alpha}} e^{-i\pi/4} e^{i\omega^2/(4\alpha)},$$

$$(TF12) \quad \mathcal{F}\left[\sqrt{\frac{\pi}{\alpha}} e^{-i\pi/4} e^{ix^2/(4\alpha)}\right](\omega) = e^{-i\alpha\omega^2},$$

$$(TF13) \quad \mathcal{F}\left[\frac{\alpha}{(x - x_0)^2 + \alpha^2} + \frac{\alpha}{(x + x_0)^2 + \alpha^2}\right](\omega) = e^{-\alpha|\omega|} \cos x_0\omega,$$

$$(TF14) \quad \mathcal{F}\left[\frac{\alpha}{(x - x_0)^2 + \alpha^2} - \frac{\alpha}{(x + x_0)^2 + \alpha^2}\right](\omega) = ie^{-\alpha|\omega|} \sin x_0\omega,$$

$$(TF15) \quad \mathcal{F}\left[\frac{1}{(x^2 + \alpha^2)(x^2 + \beta^2)}\right](\omega) = \frac{1}{2\alpha\beta(\alpha^2 - \beta^2)} (\alpha e^{-\beta|\omega|} - \beta e^{-\alpha|\omega|}),$$

$$(TF16) \quad \mathcal{F}\left[\frac{1}{x}\right](\omega) = \begin{cases} -i/2, & \text{if } \omega < 0, \\ 0, & \text{if } \omega = 0, \\ i/2, & \text{if } \omega > 0, \end{cases} \quad (\text{it's understood as the principal value}).$$