uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms

Chapter 2: Transforms

Section 2.3: Laplace transform

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3 Laplace transform

Definition 3.1 Let $\mathcal{E}$ be the class of complex functions $f : (0, \infty) \longrightarrow \mathbb{C}$ such that

(1) f is integrable on $[0, T]$ for all $T > 0$ (this condition holds, for example if f is piecewise continuous).

(2) f has <u>exponential growth</u>: $\lim_{t\to\infty} f(t)\, e^{-\alpha t} = 0$ for some $\alpha \in \mathbb{R}$.

Definition 3.2 If $f \in \mathcal{E}$, then we define its <u>Laplace transform</u> as

$$\mathcal{L}f(z) = \int_0^{\infty} f(t)\, e^{-zt} dt\,.$$

This integral converges for $\operatorname{Re} z > \alpha$.

Lemma 3.3 (Riemann-Lebesgue lemma for Laplace transform). *Let $f \in \mathcal{E}$. Then*

a) $\mathcal{L}f(x + iy) \to 0$ as $|y| \to \infty$ for each fixed $x > \alpha$.

b) $\mathcal{L}f(x + iy) \to 0$ as $x \to \infty$ for each fixed y.

Theorem 3.4 (Properties of Laplace transform). *Let $f, g \in \mathcal{E}$, $\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{R}$.*

(1) $\mathcal{L}[\alpha f + \beta g](z) = \alpha\, \mathcal{L}[f](z) + \beta\, \mathcal{L}[g](z)$.

(2) $\mathcal{L}[e^{at} f(t)](z) = \mathcal{L}[f](z - a)$.

(3) $\mathcal{L}[f(at)](z) = \dfrac{1}{a}\, \mathcal{L}[f(t)](z/a), \qquad (a > 0)$.

(4) $\mathcal{L}[f(t-a)H(t-a)](z) = e^{-az} L[f](z)$, *where* $a > 0$ *and* $H(t) = \begin{cases} 0\,, & \text{if } x \le 0\,, \\ 1\,, & \text{if } x > 0\,, \end{cases}$ *(Heaviside function).*

(5) *If f is continuous on $[0, \infty)$, f' is piecewise continuous on $(0, \infty)$, and $f, f' \in \mathcal{E}$, then*

$$\mathcal{L}f'(z) = z\, \mathcal{L}f(z) - f(0)\,.$$

(6) *If $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $(0, \infty)$, and $f, f', \ldots, f^{(n)} \in \mathcal{E}$, then*

$$\mathcal{L}[f^{(n)}](z) = z^n\, \mathcal{L}f(z) - z^{n-1} f(0) - z^{n-2} f'(0) - \cdots - z f^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

(7) *$\mathcal{L}f(z)$ is holomorphic on $\operatorname{Re} z > \alpha$ if $f(t) \in \mathcal{E}$ (with exponential growth α) and*

$$\frac{d^n}{dz^n}[\mathcal{L}f(z)] = (-1)^n\, \mathcal{L}[t^n f(t)](z)\,.$$

(8) *If $f \in \mathcal{E}$ then $g(t) = \int_0^t f(x)\, dx \in \mathcal{E}$ and $\mathcal{L}g(z) = \frac{1}{z}\, \mathcal{L}f(z)$.*

(9) *If $f(t)/t$ is integrable on $[0, T]$ for all $T > 0$, then $\mathcal{L}[\frac{f(t)}{t}](z) = \int_z^{\infty} \mathcal{L}f(z)\, dz$.*

Remark 3.5 In (9), $\int_z^{\infty}$ denotes integration over any curve in the z-plane starting at z and such that along the curve $\operatorname{Im} z$ stays bounded and $\operatorname{Re} z \to \infty$.

Definition 3.6 (Convolution for Laplace transform). If $f, g \in \mathcal{E}$ then the <u>convolution</u> of f and g is defined as

$$(f * g)(t) = \int_0^t f(x)\, g(t - x)\, dx\,.$$

Proposition 3.7 *$f * g \in \mathcal{E}$ and $\mathcal{L}[f * g](z) = \mathcal{L}f(z)\, \mathcal{L}g(z)$.*

3.1 Inverse Laplace transform

We want to solve the equation $\mathcal{L}f(z) = F(z)$ where $F(z)$ is a known function. It is clear that the solution, if exists, is not unique since changing the value of f at a countable set does not change the value of $\mathcal{L}f(z)$. However, we have

Theorem 3.8 (Lerch's theorem). *If f and g are two different continuous functions on $(0, \infty)$ such that their Laplace transforms exist, then $\mathcal{L}f \neq \mathcal{L}g$.*

Definition 3.9 Given a derivable function $F(z)$ on $\operatorname{Re} z > \alpha$, we define its *inverse Laplace transform* as the unique continuous function $f : (0, \infty) \longrightarrow \mathbb{C}$ such that $\mathcal{L}f = F$.

Theorem 3.10 (Mellin's inversion formula) *Let $F(z)$ be a derivable function on the half-plane $\operatorname{Re} z > \alpha$ such that $F = \mathcal{L}f$ with $f : (0, \infty) \longrightarrow \mathbb{C}$ continuous. Then*

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{zt} F(z)\, dz = \lim_{R\to\infty} \int_{\Gamma_R} e^{zt} F(z)\, dz$$

where Γ_R is the vertical segment $\{x + iy : \ |y| \leq R\}$ oriented from $x - iR$ to $x + iR$, and $x > \alpha$.

LAPLACE TRANSFORMS TABLE

$$f(t) = 1\,, \qquad \mathcal{L}[f](z) = \frac{1}{z} \qquad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^n \quad (n \in \mathbb{N})\,, \qquad \mathcal{L}[f](z) = \frac{n!}{z^{n+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t^a \quad (a > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(a+1)}{z^{a+1}} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2+a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2+a^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = \frac{\sin at}{t} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \arctan\frac{a}{z} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = e^{at} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{1}{z-a} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}t^b \quad (a \in \mathbb{R},\, b > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(b+1)}{(z-a)^{b+1}} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}\sin bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{b}{(z-a)^2+b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = e^{at}\cos bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z-a}{(z-a)^2+b^2} \quad (\operatorname{Re} z > a)\,,$$

$$f(t) = \sinh at = \frac{e^{at}-e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2-a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \cosh at = \frac{e^{at}+e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2-a^2} \quad (\operatorname{Re} z > |a|)\,,$$

$$f(t) = \sin at - at\cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2a^3}{(z^2+a^2)^2} \quad (\operatorname{Re} z > 0)\,,$$

$$f(t) = t\sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2az}{(z^2+a^2)^2} \quad (\operatorname{Re} z > 0)\,.$$