uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.2: Holomorphic functions

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1.2. HOLOMORPHIC FUNCTIONS

2.1. At what points are the following functions differentiable in complex sense?

$$
\begin{aligned}
&a)\ f(z) = |z|\,, \\
&b)\ f(z) = |z|^2\,, \\
&c)\ f(z) = |z|^\alpha\,, \quad \alpha > 0\,, \\
&d)\ f(z) = \sqrt{|xy|}\,, \quad z = x + iy\,, \\
&e)\ f(z) = \overline{z}\,, \\
&f)\ f(z) = h(x)\,, \quad h \in C^1(\mathbf{R}, \mathbf{C})\,, \\
&g)\ f(z) = h(y)\,, \quad h \in C^1(\mathbf{R}, \mathbf{C})\,, \\
&h)\ f(z) = h(x, y)\,, \quad h \in C^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})\,, \\
&i)\ f(z) = \frac{1}{z^2} + |z|^2\,, \\
&j)\ f(z) = z \operatorname{Re} z\,, \\
&k)\ f(z) = \frac{1}{(z + 1/z)^2}\,, \\
&l)\ f(z) = e^z := e^x \cos y + i e^x \sin y\,, \\
&m)\ f(z) = e^{z^3/(z+3)}.
\end{aligned}
$$

2.2. Find a holomorphic function f with real part:

$$
\begin{aligned}
&a)\ u(x, y) = ax + by + c\,, \qquad a, b, c \in \mathbf{R}\,, \\
&b)\ u(x, y) = e^{-x}(x \sin y - y \cos y)\,, \\
&c)\ u(x, y) = e^{-y} \cos x\,, \\
&d)\ u(x, y) = \frac{x}{x^2 + y^2}\,, \\
&e)\ u(x, y) = e^{x^2 - y^2} \sin 2xy\,, \\
&f)\ u(x, y) = x\, e^{-x} \cos y + y\, e^{-x} \sin y\,, \\
&g)\ u(x, y) = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy\,, \\
&h)\ u(x, y) = \log \sqrt{x^2 + y^2}\,, \\
&i)\ u(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}\,, \\
&j)\ u(x, y) = \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} x^{2n-2k} y^{2k}\,.
\end{aligned}
$$

In which domain is f holomorphic in each case?

2.3. a) Prove that a harmonic function u on an open connected set U has a harmonic conjugate if and only if there exists a holomorphic function f on U such that $f' = u_x - i u_y$.

b) Let $z_0 = x_0 + iy_0$ and $u \in \mathbf{C}^2(D(z_0, r))$ be a harmonic function. Prove that

$$
v(x_1, y_1) = c + \int_{y_0}^{y_1} \frac{\partial u}{\partial x}(x_1, y)\, dy - \int_{x_0}^{x_1} \frac{\partial u}{\partial y}(x, y_0)\, dx
$$

is a harmonic conjugate of u on $D(z_0, r)$ with $v(x_0, y_0) = c$.

Hint: You can use the Fundamental Theorem of Calculus in order to compute the partial derivatives of $v(x_1, y_1)$.

2.4. Prove that the Cauchy-Riemann equations on polar coordinates are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\,, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}\,.$$

2.5. Using if required the Cauchy-Riemann equations on polar coordinates, find the holomorphic function verifying:

$$\begin{aligned} &a)\text{ its modulus is } e^{r^2\cos 2\theta}\,,\\ &b)\text{ its modulus is } (x^2+y^2)\,e^x\,,\\ &c)\text{ its argument is } xy\,,\\ &d)\text{ its argument is } \theta + r\,\sin\theta\,,\\ &e)\text{ its real part is } \frac{\cos\theta+\sin\theta}{r}\,. \end{aligned}$$

In which domain is f holomorphic in each case?

Hint: $a)$ and $c)$: if $p(z)$ is a polynomial, then $e^{p(z)}$ is a holomorphic function.

2.6. If $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, with $a, b, c, d \in \mathbf{R}$, for what values of a, b, c, d is u harmonic?

2.7. Find the harmonic functions which can be written as:

$$\begin{array}{lll} a)\ g(ax+by)\,,\quad a,b\in\mathbf{R}\,, & b)\ g(x^2+y^2)\,, & c)\ g(xy)\,,\\ d)\ g(x+\sqrt{x^2+y^2}\,)\,, & e)\ g((x^2+y^2)/x)\,, & f)\ g(y/x)\,, \end{array}$$

with g a real function of class C^2. Find in each case the holomorphic functions whose real parts are these functions.

2.8. Find a holomorphic function f on $\mathbf{C}$ such that $f(1+i) = 0$, $f'(0) = 0$ and $\operatorname{Re} f'(z) = 3\,(x^2 - y^2) - 4y$.

2.9. Prove that if u and v are harmonic functions and $\alpha, \beta \in \mathbf{R}$, then $\alpha u + \beta v$ is harmonic. For what harmonic functions u is also harmonic the function u^2?

Hint: $\Delta(u^2) = 2|\nabla u|^2 + 2u\Delta u$.

2.10. Let U be an open connected subset of $\mathbf{C}$ and $f : U \longrightarrow \mathbf{C}$ a holomorphic function. Prove the following statements:

a) If $f'(z) = 0$ on U, then f is constant on U.
b) If $\operatorname{Re} f$ is constant on U, then f is constant on U.
c) If $\operatorname{Im} f$ is constant on U, then f is constant on U.
d) If $|f|$ is constant on U, then f is constant on U.
e) If $\overline{f}$ is holomorphic on U, then f is constant on U.

2.11. Assume that f is holomorphic on $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Study if each function g below is holomorphic on $\mathbf{D}$:

$$a)\ g(z) = \overline{f(z)}\,, \qquad b)\ g(z) = f(\overline{z})\,, \qquad c)\ g(z) = \overline{f(\overline{z})}\,, \qquad d)\ g(z) = |f(z)|\,, \qquad e)\ g(z) = f(z)\,f(\overline{z})\,.$$

Do we get the same result if we replace $\mathbf{D}$ by any domain A? And if A satisfies that $z \in A \Leftrightarrow \overline{z} \in A$?

2.12. Prove that if $h : \mathbf{R}^2 \longrightarrow \mathbf{R}$ is of class C^2 and f is holomorphic, then

$$\Delta(h \circ f) = \Delta h(f) \cdot |f'|^2 .$$

2.13. Prove that $f(z) = z^2$ is holomorphic on $\mathbf{C}$ by checking that if satisfies the Cauchy-Riemann equations. If $f(z_0) = a + bi$, prove that the curves $\operatorname{Re} f(z) = a$ and $\operatorname{Im} f(z) = b$ are orthogonal at z_0.

Hint: Prove that the inner product of the tangent vectors of the two curves is 0.

2.14. Find all the holomorphic functions which can be written as $u(x) + i\,v(y)$.

2.15. a) If $Q(z)$ is a polynomial with different roots $z_1, \ldots, z_n$, and if $P(z)$ is a polynomial of degree less than n, prove that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(z_k)}{Q'(z_k)\,(z - z_k)} .$$

b) Prove that there exists a unique polynomial p of degree less than n with $p(z_k) = w_k$ (*Lagrange's interpolation formula*).

Hints: a) $P(z)$ and $\sum_{k=1}^{n} \frac{P(z_k)Q(z)}{Q'(z_k)\,(z-z_k)}$ have the same values at $z_1, \ldots, z_n$. b) Use the formula in the previous item.