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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.7: Laurent series

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1.7. LAURENT SERIES

7.1. Compute the power series about 0 of the following functions and find the radius of convergence in each case:

$$a)\ \sin^2 z\,,\qquad b)\ \cosh^2 z\,,\qquad c)\ \sqrt{z+i}\,,\qquad d)\ \frac{6z}{z^2-4z+13}\,,\qquad e)\ \frac{z^2}{(z+1)^2}\,,$$

$$f)\ \log\frac{1+z}{1-z}\,,\qquad g)\ \arctan z\,,\qquad h)\ \arcsin z\,,\qquad i)\ \operatorname{arsinh} z\,,\qquad j)\ \sin z^2\,,$$

$$k)\ \frac{\sin z}{z}\,,\qquad l)\ \frac{e^z-1-z}{z^2}\,,\qquad m)\ \frac{1-\cos z^3}{z^5}\,,\qquad n)\ (1-z)^{-p}\quad (p\in\mathbf{C})\,,\qquad o)\ \int_0^1 e^{t^2z^2}\,dt\,,$$

7.2. Compute the power series about 1 of the following functions and find the radius of convergence in each case:

$$a)\ \frac{z^2}{(z+1)^2}\,,\qquad b)\ \frac{4z}{z^2-2z+5}\,,\qquad c)\ z^{1/3}\,,\qquad d)\ \sin(2z-z^2)\,.$$

7.3. Compute the terms with degree less than three in the power series about 0 of the following functions (without compute their derivatives):

$$a)\ (\log(1-z))^2\,,\qquad b)\ e^{z/(1-z)}\,,\qquad c)\ e^{e^z}\,,\qquad d)\ (1+z)^z\,,$$

$$e)\ e^{z\,\sin z}\,,\qquad f)\ \sqrt{\cos z}\,,\qquad g)\ \frac{z}{e^z-1}\,.$$

7.4. a) Let $f(z)=\sum_{n=0}^{\infty} a_n\,(z-a)^n$ for $|z-a|<R$. Prove the *Parseval's formula*:

$$\int_0^{2\pi} |f(a+re^{it})|^2\,dt = 2\pi\sum_{n=0}^{\infty}|a_n|^2\,r^{2n}\,,\qquad \text{if } 0\le r<R\,.$$

Hint: $|f(a+re^{it})|^2 = (\sum_{n=0}^{\infty} a_n r^n e^{itn})(\sum_{m=0}^{\infty}\overline{a_m} r^m e^{-itm})$.

b) If $M(r)=\max\{|f(z)| : |z-a|=r\}$, prove

$$\sum_{n=0}^{\infty}|a_n|^2\,r^{2n}\le M(r)^2\,,\qquad \text{if } 0\le r<R\,.$$

Hint: Use the previous item.

c) Prove that we have for every $n\in\mathbf{N}$

$$|a_n|\le\frac{M(r)}{r^n}\,,\qquad \text{if } 0<r<R\,.$$

Hint: Use item b).

d) Assume that there exist $n\in\mathbf{N}$ and $0<r<R$ such that $|a_n|=M(r)/r^n$. Prove that $f(z)=a_n(z-a)^n$.

Hint: Use item b).

7.5. Compute the Laurent series about $z_0=0$ and $z_0=\infty$ (if it is possible) of the following functions, and find the annuli in which they are convergent:

$$a)\ \frac{1}{z^2-3z+2}\,,\qquad b)\ \frac{1}{z}+\frac{1}{(z-1)^2}+\frac{1}{z+2}\,,$$

$$c)\ \sin\frac{1}{z}\,,\qquad d)\ \frac{1}{z(1-z)}\,,\qquad e)\ \frac{1}{(z-a)^k}\quad (a\ne 0, k\in\mathbf{N})\,,$$

$$f)\ z^2e^{1/z}\,,\qquad g)\ \log\frac{z-a}{z-b}\,,\qquad h)\ \frac{\arctan z}{z^4}\,,$$

$$i)\ \frac{1}{(z-a)(z-b)}\,,\quad \text{in terms of the values of } a,b\in\mathbf{C}.$$

7.6. Compute the Laurent series about the indicated points of the following functions, and find the annuli in which they are convergent:

$$a)\ \frac{1}{(z^2+1)^2}\,,\qquad z_0=i,\qquad b)\ e^{1/(1-z)}\,,\qquad z_0=1,$$
$$c)\ \frac{\sin z}{(z-\pi)^2}\,,\qquad z_0=\pi,\qquad d)\ z^2\sin\frac{1}{z-1}\,,\qquad z_0=1,$$
$$e)\ \sin\frac{z}{1-z}\,,\qquad z_0=1,\qquad f)\ \cos\frac{z^2-4z}{(z-2)^2}\,,\qquad z_0=2.$$

7.7. Compute the Laurent series on the indicated regions Ω of the following functions:

$$a)\ \frac{1}{(z-a)(z-b)}\,,\quad (0<|a|<|b|)\,,\ \Omega=\{z:|a|<|z|<|b|\}\,,$$
$$b)\ \frac{z^2-2z+5}{(z-2)(z^2+1)}\,,\quad \Omega=\{z:1<|z|<2\}\,,$$
$$c)\ \frac{1}{(z-2)^2}\,,\quad \Omega_1=\{z:|z|<2\}\,,\ \Omega_2=\{z:|z|>2\}\,,$$
$$d)\ e^{z+1/z}\,,\quad \Omega=\{z:0<|z|<\infty\}\,,$$
$$e)\ \sin z\,\sin\frac{1}{z}\,,\quad \Omega=\{z:0<|z|<\infty\}\,.$$

7.8. L'Hôpital-Bernoulli's rule for complex functions:

a) Let f and g be two analytic functions with a zero of order k at z_0. Prove that f/g has a removable singularity at z_0 and

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}\,.$$

b) Let $f(z)$ and $g(z)$ be analytic functions at z_0 such that $f(z_0)\neq 0$, $g(z_0)=0$ and $g'(z_0)\neq 0$. Prove that $h=f/g$ has a simple pole at z_0 and that $\mathrm{Res}\,(h,z_0)=f(z_0)/g'(z_0)$.

c) Let $f(z)$ and $g(z)$ be analytic functions at z_0 which have at z_0 a zero of order k and $k+1$, respectively. Prove that $h=f/g$ has a simple pole at z_0 and $\mathrm{Res}\,(h,z_0)=(k+1)\,f^{(k)}(z_0)/g^{(k+1)}(z_0)$.

d) Let f and g be analytic functions at z_0 which have a zero at z_0 of order n and k, respectively, with $k\geq n$. Prove that $h=f/g$ has a pole at z_0 of order $k-n$. Find a formula (as in item b)) for the residue of h on z_0.

e) Think what happens on each one of the previous cases if f and/or g have at z_0 a pole instead of a zero.

Hint: Write the Taylor (or Laurent) series of f and g.

7.9. Determine the nature of the singularities of each of the following functions (including the point ∞, that is, the singularity of $f(1/z)$ at $z=0$) and compute the residue at the (finite) singularities:

$$a)\ \frac{e^z}{1+z^2}\,,\qquad b)\ z\,e^{-z}\,,\qquad c)\ \frac{e^z}{z(1-e^{-z})}\,,\qquad d)\ \frac{z-\sin z}{z^3}\,,$$
$$e)\ (z-3)\sin\frac{1}{z+2}\,,\qquad f)\ \frac{1-\cos z}{z}\,,\qquad g)\ \frac{z^2}{e^{z^4}}\,,\qquad h)\ \frac{1}{z}\cosh\frac{1}{z}\,,$$
$$i)\ \cos\left(z^2+\frac{1}{z^2}\right)\,,\qquad j)\ e^{z/(z-2)}\,,\qquad k)\ e^{-z}\cos\frac{1}{z}\,,\qquad l)\ \frac{1}{z^2}+\sin\frac{1}{z}\,,$$
$$m)\ \frac{e^{2z}}{(z-1)^3}\,,\qquad n)\ \frac{z^{2n}}{(1+z)^n}\,,\qquad o)\ \frac{\mathrm{cotan}\,z\,\mathrm{cotanh}\,z}{z^3}\,,\qquad p)\ \mathrm{cosec}\,\frac{1}{z}\,,$$
$$q)\ \frac{z^4}{1+z^4}\,,\qquad r)\ \frac{1}{z^3-z^5}\,,\qquad s)\ \frac{1-e^z}{1+e^z}\,,$$
$$t)\ \frac{1}{\sin z-\sin a}\,,\qquad u)\ z\,e^{z/(1-z)}-\frac{1}{(z-1)^2}\sin\frac{1}{z}\,,\qquad v)\ \frac{\tan z}{z^n}\,.$$

7.10. The *Bernoulli numbers* B_n are defined by the power series of $z/(e^z-1)$:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}\,z^n\,.$$

Prove that Bernoulli numbers satisfy:

$$a)\quad \binom{n+1}{0}B_0+\binom{n+1}{1}B_1+\cdots+\binom{n+1}{n}B_n=0\,,\qquad \text{for } n\geq 1$$

$$b)\quad B_0=1\,,\ B_1=-\frac{1}{2}\,,\ B_2=\frac{1}{6}\,,\ B_4=-\frac{1}{30}\,,\ B_6=\frac{1}{42}\,,\ B_8=-\frac{1}{30}\,,$$

$$c)\quad B_{2n+1}=0\,,\qquad \text{for } n\geq 1\,.$$

d) Prove that $\limsup_{n\to\infty}\left(\frac{|B_n|}{n!}\right)^{1/n}=\frac{1}{2\pi}\,.$

Hint: Find the distance from 0 to the closest singularity of the function $z/(e^z-1)$.

7.11. Compute the power series about 0 of each of the following functions, by applying the previous exercise, and find the radius of convergence in each case:

$$a)\quad \frac{z}{e^z+1}\,,\qquad \left(\text{prove first that } \frac{z}{e^z+1}=\frac{z}{e^z-1}-\frac{2z}{e^{2z}+1}\right)$$

$$b)\quad \tan z\,,\qquad \left(\text{prove first that } \tan z=-i+\frac{2i}{e^{2iz}+1}\right)$$

$$c)\quad z\,\mathrm{cotan}\,z\,,\qquad \left(\text{prove first that } z\,\mathrm{cotan}\,z=iz+\frac{2iz}{e^{2iz}-1}\right)$$

$$d)\quad \frac{z}{\sin z}\,,\qquad \left(\text{prove first that } \frac{z}{\sin z}=\frac{iz}{e^{iz}-1}+\frac{iz}{e^{iz}+1}\right)$$

$$e)\quad \log\cos z\,,\qquad \left(\text{prove first that } (\log\cos z)'=-\tan z\right)$$

$$f)\quad \log\frac{\tan z}{z}\,,\qquad \left(\text{prove first that } \left(\log\frac{\tan z}{z}\right)'=2\,\mathrm{cosec}\,(2z)-\frac{1}{z}\right)$$

$$g)\quad \log\frac{z}{\sin z}\,,\qquad \left(\text{prove first that } \left(\log\frac{z}{\sin z}\right)'=\frac{1}{z}-\mathrm{cotan}\,z\right)$$