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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 1: Complex variable

Section 1.8: Residue theorem

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1.8. RESIDUE THEOREM

8.1. Compute the following integrals by using the residue theorem:

a) $\displaystyle\int_\gamma \frac{dz}{z^2-1}$, with $\gamma(t) = 2\,e^{it}$, $t \in [0, 2\pi]$.

b) $\displaystyle\int_\gamma \frac{dz}{z^4+1}$, where γ is the boundary of the domain $\{x+iy : x^2+y^2 < 4,\, x > 0\}$.

c) $\displaystyle\int_\gamma \frac{dz}{z^4+1}$, where γ is the circumference centered at 0 with radius 2.

d) $\displaystyle\int_\gamma \frac{1+z}{1-\cos z}\, dz$, where γ is the circumference centered at 0 with radius 7.

e) $\displaystyle\int_\gamma \frac{2+3\,\sin \pi z}{z(z-1)^2}\, dz$, where γ is the square with vertices $3+3i$, $3-3i$, $-3+3i$ and $-3-3i$.

f) $\displaystyle\int_\gamma e^{-1/z} \sin(1/z)\, dz$, where γ is the unit circumference.

g) $\displaystyle\int_\gamma \frac{dz}{(z^2+1)(z-1)^2}$, where γ is the circumference centered at $1+i$ with radius $\sqrt{2}$.

h) $\displaystyle\int_\gamma \frac{z^2\, dz}{e^{2\pi i z^3}-1}$, where γ is the circumference centered at 0 with radius r, where $n < r^3 < n+1$ for some $n \in \mathbf{N}$.

i) $\displaystyle\int_\gamma \cosh z \,\mathrm{cotan}\, z\, dz$, where γ is the circumference centered at 0 with radius $(n+1/2)\pi$.

j) $\displaystyle\int_\gamma \frac{\sin(a/z)\, dz}{z^2+b}$ and $\displaystyle\int_\gamma \frac{\cos(a/z)\, dz}{z^2+b}$, where γ is the circumference centered at 0 with radius r, $a, b \in \mathbf{C} \setminus \{0\}$, $|b| \neq r^2$.

k) $\displaystyle\int_\gamma \frac{z^2+z^{-2}}{(\overline{z}-a)(b-\overline{z})}\, dz$, where γ is the circumference centered at 0 with radius r and $0 < |a| < r < |b|$.

l) $\displaystyle\int_\gamma \Big(\sum_{n=-1}^{\infty} z^n \Big)\, dz$, where γ is the circumference centered at 0 with radius $1/2$.

m) $\displaystyle\int_\gamma z^n\, e^{1/z}\, dz$, where $n \in \mathbf{N}$ and γ is any circumference surrounding 0.

n) $\displaystyle\int_\gamma (1+z+z^2)\,(e^{1/z} + e^{1/(z-1)} + e^{1/(z-2)})\, dz$, where γ is the circumference centered at 0 with radius 3.

o) $\displaystyle\int_\gamma P(z)\,(e^{1/z} + \cdots + e^{1/(z-k)})\, dz$, where γ is the circumference centered at 0 with radius $r > k$ and $P(z)$ is a polynomial of degree k.

p) $\displaystyle\int_\gamma \frac{dz}{|z-a|^2}$, where γ is the circumference centered at 0 with radius r, and $a \in \mathbf{C}$ with $|a| \neq r$.

Hints: k) $z\overline{z} = r^2$ if $z \in \gamma$, p) $|z-a|^2 = (z-a)(\overline{z}-\overline{a}) = (z-a)(r^2/z - \overline{a})$ if $z \in \gamma$.

8.2. Compute the following integrals:

$$a)\ \int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx\,, \qquad b)\ \int_0^\infty \frac{x^2}{1+x^4}\,dx\,, \qquad c)\ \int_0^\infty \frac{dx}{x^6+1}\,,$$

$$d)\ \int_{-\infty}^\infty \frac{dx}{x^6+a^6}\,, \quad (a>0), \qquad e)\ \int_{-\infty}^\infty \frac{\cos x}{x^2-2x+2}\,dx\,, \qquad f)\ \int_{-\infty}^\infty \frac{\sin x}{x^2-2x+2}\,dx\,,$$

$$g)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2+1}\,dx\,, \qquad h)\ \int_{-\infty}^\infty \frac{\cos ax}{x^2+b^2}\,dx\,, \quad a,b>0, \qquad i)\ \int_{-\infty}^\infty \frac{\cos(\pi x/2)}{x^2-1}\,dx\,,$$

$$j)\ \text{p.v.}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx\,, \qquad k)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\,dx\,, \qquad l)\ \int_{-\infty}^\infty \frac{\sin^3 x}{x^3}\,dx\,,$$

$$m)\ \int_{-\infty}^\infty \frac{\sin^2 x}{x^2(1+x^2)}\,dx\,, \qquad n)\ \text{p.v.}\int_{-\infty}^\infty \frac{a\cos x + x\sin x}{x^2+a^2}\,dx\,, \quad a>0,$$

$$o)\ \int_0^\infty \frac{x^p}{x^n+1}\,, \quad n,\,p\in\mathbf{N},\ p<n-1, \qquad p)\ \int_0^\infty \frac{dx}{(a+bx^2)^n}\,, \quad n\in\mathbf{N},\ a,b>0\,.$$

Hints: $k)$ Integrate the function $(1-e^{2ix})/x^2$.
$l)$ Integrate the function $(-e^{3ix}+3e^{ix}-2)/x^3$.
$m)$ Integrate the function $(1-e^{2ix})/(x^2(1+x^2))$.

8.3. Compute the following integrals, if $a>0$ and $p\in(-1,1)$:

$$a)\ \int_0^\infty \frac{\log x}{x^2+a^2}\,dx\,, \qquad b)\ \int_0^\infty \frac{\sqrt{x}\log x}{x^2+a^2}\,dx\,, \qquad c)\ \int_0^\infty \frac{x^p}{x^2+a^2}\,dx\,, \qquad d)\ \int_0^\infty \frac{x^p\log x}{x^2+a^2}\,dx\,,$$

$$e)\ \int_0^\infty \frac{\log^2 x}{x^2+a^2}\,dx\,, \qquad f)\ \int_0^\infty \frac{\sqrt{x}\log^2 x}{x^2+a^2}\,dx\,, \qquad g)\ \int_0^\infty \frac{x^p\log^2 x}{x^2+a^2}\,dx\,,$$

$$h)\ \int_0^\infty \frac{\log x}{(x+a)^2}\,dx\,, \qquad i)\ \int_0^\infty \frac{\sqrt{x}\log x}{(x+a)^2}\,dx\,, \qquad j)\ \int_0^\infty \frac{x^p}{(x+a)^2}\,dx\,, \qquad k)\ \int_0^\infty \frac{x^p\log x}{(x+a)^2}\,dx\,,$$

$$l)\ \int_0^\infty \frac{\log^2 x}{(x+a)^2}\,dx\,, \qquad m)\ \int_0^\infty \frac{\sqrt{x}\log^2 x}{(x+a)^2}\,dx\,, \qquad n)\ \int_0^\infty \frac{x^p\log^2 x}{(x+a)^2}\,dx\,,$$

$$o)\ \int_0^\infty \frac{x\log^3 x}{(x+a)^2}\,dx\,, \qquad p)\ \int_0^\infty \frac{x^{-p}}{x+a}\,dx\,, \qquad q)\ \int_0^\infty \frac{x^{-p}\log x}{x+a}\,dx\,.$$

Hints: $a)$, $b)$, $c)$, $d)$, $e)$, $f)$, $g)$, integrate along the boundary curve of the domain $\{z\in\mathbf{C}:\ \varepsilon<|z|<R,\ \operatorname{Im} z>0\}$. For the other items, you can integrate along the boundary curve of the domain $\{z\in\mathbf{C}:\ \varepsilon<|z|<R\}\setminus\{x+iy\in\mathbf{C}:\ x\ge 0,\ |y|\le\delta\}$. $h)$ integrate the function $(\log^2 z)/(z+a)^2$. $l)$ integrate the function $(\log^3 z)/(z+a)^2$. $o)$ is it an integrable function?

8.4. Compute the following integrals:

$$a)\ \int_{-\infty}^\infty \frac{\sin(ax)}{\sinh x}\,dx\,, \quad a\in\mathbf{R} \qquad b)\ \int_{-\infty}^\infty \frac{\cos(ax)}{\cosh x}\,dx\,, \quad a\in\mathbf{R} \qquad c)\ \int_{-\infty}^\infty \frac{e^{ax}}{\cosh x}\,dx\,, \quad a\in(-1,1)\,,$$

$$d)\ \int_0^\infty \frac{\cosh(ax)}{\cosh x}\,dx\,, \quad a\in(-1,1)\,, \qquad e)\ \int_0^\infty \frac{\cosh(ax)}{\cosh(\pi x)}\,dx\,, \quad a\in(-\pi,\pi)\,,$$

$$f)\ \int_0^\infty \frac{x^2}{\cosh x}\,dx\,, \qquad g)\ \int_0^\infty \frac{x\cos(ax)}{\sinh x}\,dx\,.$$

Hints: $a)$ $\sinh(z+2\pi i)=\sinh z$. $b)$ $\cosh(z+\pi i)=-\cosh z$. $c)$ Use item $b)$. $d)$ Use item $c)$. $e)$ Use item $d)$. $f)$ Use item $d)$ and integrate the function $x^3/\cosh x$.

8.5. Using that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, prove that

$$\int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\,dx = \sqrt{\pi}\, e^{-b^2}\,.$$

Hint: Integrate the function e^{-z^2} along an appropriate closed curve.

8.6. Compute the following integrals, if $a,\, b > 0$:

$$a)\ \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\,\sin\theta}\,,\ a^2 > b^2 + c^2\,,\quad b)\ \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}\,,\ a > b > 0\,,\quad c)\ \int_0^{2\pi} \frac{d\theta}{(a + b\cos^2\theta)^2}\,,$$

$$d)\ \int_0^{2\pi} e^{2\cos\theta}\,d\theta\,,\qquad e)\ \int_0^{2\pi} \frac{\cos(n\theta + \varphi)}{1 - 2a\cos\theta + a^2}\,d\theta\,,\quad a > 1,\ n \in \mathbf{N}\,,\qquad f)\ \int_0^{2\pi} \cos^n\theta\,d\theta\,,\ n \in \mathbf{N}\,.$$

8.7. Prove that

$$\int_0^{\pi} e^{\cos\theta} \cos(\sin\theta)\,d\theta = \pi.$$

Hint: Use the complex integral

$$\int_{|z|=1} \frac{e^z}{z}\,dz\,.$$

8.8. Prove the following equalities, if $a \in \mathbf{R}$:

$$a)\ \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2(\pi a)}\,,\qquad b)\ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,,\qquad c)\ \sum_{n=0}^{\infty} \frac{1}{a^2+n^2} = \frac{1}{2a^2}(1 + \pi a \,\mathrm{cotanh}\,(\pi a))\,,$$

$$d)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{1}{2a^2}\Big(1 + \frac{\pi a}{\sinh(\pi a)}\Big)\,,\qquad e)\ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}\,,$$

$$f)\ \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k-1} \pi^{2k} B_{2k}}{(2k)!}\,,\qquad g)\ \sum_{n=1}^{\infty} \frac{1}{n^3}\,.$$

Remark: The last item is very complicated.

Hint: $f)$ Use the Laurent series $\pi\,\mathrm{cotan}\,\pi z = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \pi^{2n} B_{2n} z^{2n-1}/(2n)!$

8.9. Compute the Fresnel integrals

$$\int_0^{\infty} \cos x^2\,dx = \frac{\sqrt{2\pi}}{4}\,,\qquad \int_0^{\infty} \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}\,,$$

which appear in diffraction theory.

Hint: Integrate the function $f(z) = e^{iz^2}$ along the boundary curve of the circular sector $\{z \in \mathbf{C} :\, |z| < R,\ \arg z \in (0, \pi/4)\}$. Use $\int_0^{\infty} e^{-x^2}\,dx = \sqrt{\pi}/2$ and prove the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$.

8.10. For each $z \in \mathbf{C}$ with $\mathrm{Re}\, z > 0$, the Gamma function $\Gamma(z)$ is defined as $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\,dx$.

a) Prove that, for each $0 < a < b$, there exists an integrable function $f : (0, \infty) \to \mathbf{R}$ such that $|x^{z-1} e^{-x}| \leq f(x)$ for every $x > 0$ and z with $a \leq \mathrm{Re}\, z \leq b$.

Hint: Consider separately the cases $0 < x \leq 1$ and $x > 1$.

b) Prove that Γ is holomorphic on $\{\mathrm{Re}\, z > 0\}$.

Hint: Use Morera's theorem.

c) Prove that $\Gamma(z+1) = z\Gamma(z)$ on the set $\{z \in \mathbf{C} :\, \mathrm{Re}\, z > 0\}$.

Hint: Use integration by parts.

d) Prove that we have, for every $n \in \mathbf{N}$, $\Gamma(z+n+1) = (z+n)(z+n-1)\cdots(z+1)z\Gamma(z)$ on $\{\mathrm{Re}\, z > 0\}$. Deduce that $\Gamma(n+1) = n!$ for every $n \in \mathbf{N}$.

e) By using the previous item, for each z on $\{\mathrm{Re}\, z > -n-1\}$, we can define $\Gamma(z)$ as

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)\cdots(z+1)z} .$$

Prove that Γ is well defined on $\mathbf{C}$, i.e., if $m > n$ and z satisfies $\mathrm{Re}\, z > -n-1$, then the corresponding definitions of $\Gamma(z)$ to n and m are the same.

f) The Gamma function is defined on the whole complex plane by the previous item. Prove that it is a meromorphic function on $\mathbf{C}$, and its unique poles are $\{0, -1, -2, \dots\}$. Prove also that each pole is simple and $\mathrm{Res}\,(\Gamma, -n) = (-1)^n/n!$.

Hint: Compute the limit $\lim_{z\to -n}(z+n)\Gamma(z)$.

8.11. a) The Beta function $B(z,w)$ is defined as

$$B(z,w) = \int_0^1 x^{z-1}\,(1-x)^{w-1}\,dx .$$

Prove that $x^{z-1}\,(1-x)^{w-1}$ is an integrable function on $(0,1)$ for any complex numbers z, w with $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, w > 0$, and that $B(z,w) = B(w,z)$.

b) Prove that this function is: b.1) holomorphic in the variable z on the domain $\{z : \mathrm{Re}\, z > 0\}$ for each w with $\mathrm{Re}\, w > 0$; b.2) holomorphic in the variable w on the domain $\{w : \mathrm{Re}\, w > 0\}$ for each z with $\mathrm{Re}\, z > 0$.

Hint: Use Morera's theorem and the symmetry of the Beta function.

c) Prove that

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)} ,$$

for any complex numbers z, w with $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, w > 0$.

Hint: Prove that $\Gamma(z)\,\Gamma(w) = 4\int_0^\infty \int_0^\infty e^{-s^2-t^2} s^{2z-1} t^{2w-1}\, ds\, dt$ and use polar coordinates.

d) If $0 < a < 1$, prove that

$$\int_0^\infty \frac{x^{-a}}{x+1}\, dx = \frac{\pi}{\sin(a\pi)} .$$

e) By making a change of variable in the integral in the previous item, prove the formula:

$$B(z, 1-z) = \frac{\pi}{\sin(\pi z)} , \qquad 0 < z < 1 .$$

Hint: Make the change of variable $t = 1/(x+1)$.

f) Prove that this formula holds for every $z \in \mathbf{C} \setminus \mathbf{Z}$, if we define $B(z,w)$ by the formula in item c) as a meromorphic function on the whole plane (on each variable).

Hint: If two holomorphic functions f, g on a domain Ω verify $f = g$ on a real interval, then $f = g$ on Ω (why?).

g) Prove that, for $b > a > -1$,

$$\int_0^{\pi/2} \cos^a t\, \cos bt\, dt \;=\; \frac{\pi\Gamma(a+1)}{2^{a+1}\,\Gamma(1+(a+b)/2)\,\Gamma(1+(a-b)/2)} .$$

Hint: Consider the integral $\int_\gamma (z+1/z)^a z^{b-1}\, dz$, where γ is the boundary of the domain $\{z \in \mathbf{C} : \mathrm{Re}\, z > 0,\ \varepsilon < |z| < 1,\ \varepsilon < |z-i|,\ \varepsilon < |z+i|\,\}$.

h) Does this formula hold if $a, b \in \mathbf{C}$ and $\mathrm{Re}\, a > -1$?

Hint: If two holomorphic functions f, g on a domain Ω verify $f = g$ on a real interval, then $f = g$ on Ω.