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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.1: Fourier series

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1 Fourier series

Problem 1.1 Given $k \geq 1$ and $a \in \mathbb{R}$, compute the Fourier series of the following functions defined on $[-L, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = 1\,, \qquad \mathbf{2.}\ f(x) = \cos\frac{k\pi x}{L}\,, \\ &\mathbf{3.}\ f(x) = x\,, \qquad \mathbf{4.}\ f(x) = x^2 \\ &\mathbf{5.}\ f(x) = e^{ax}, \qquad \mathbf{6.}\ f(x) = x^{-2}, \\ &\mathbf{7.}\ f(x) = \frac{R^2 - r^2}{R^2 - 2Rr\cos x + r^2}\,, \quad \text{with } L = \pi \text{ and } 0 < r < R\,. \end{aligned}$$

Problem 1.2 Given $a \in \mathbb{R}$, compute the Fourier cosine series of the following functions defined on $[0, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = 1\,, \qquad \mathbf{2.}\ f(x) = e^{ax}, \\ &\mathbf{3.}\ f(x) = x\,, \qquad \mathbf{4.}\ f(x) = x^2. \end{aligned}$$

Problem 1.3 Given $a \in \mathbb{R}$, compute the Fourier sine series of the following functions defined on $[0, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = 1\,, \qquad \mathbf{2.}\ f(x) = e^{ax}, \\ &\mathbf{3.}\ f(x) = x\,, \qquad \mathbf{4.}\ f(x) = x^2. \end{aligned}$$

Problem 1.4 Given $a \in \mathbb{R}$, compute the Fourier series of the following functions defined on $[-L, L]$:

$$\begin{aligned} &\mathbf{1.}\ f(x) = |x|\,, \qquad \mathbf{2.}\ f(x) = x|x|\,, \\ &\mathbf{3.}\ f(x) = e^{a|x|}, \qquad \mathbf{4.}\ f(x) = \frac{x}{|x|}\,. \end{aligned}$$

Hint: Use the two previous problems.

Problem 1.5 Prove the equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

Hint: Apply (FSe) to the function in (FS4).

Problem 1.6 Prove that if f is continuous, f' is piecewise continuous on $[-L, L]$ and $f(-L) = f(L)$, then

$$a_0(f') = 0\,, \quad a_n(f') = \frac{n\pi}{L}\, b_n(f) \quad \text{and} \quad b_n(f') = \frac{-n\pi}{L}\, a_n(f)\ (n \geq 1)\,.$$

Problem 1.7 Prove that if f and f' are continuous, f'' is piecewise continuous on $[-L, L]$, $f(-L) = f(L)$, and $f'(-L) = f'(L)$, then

$$a_0(f'') = 0\,, \quad a_n(f'') = -\left(\frac{n\pi}{L}\right)^2 a_n(f) \quad \text{and} \quad b_n(f'') = -\left(\frac{n\pi}{L}\right)^2 b_n(f)\ (n \geq 1)\,.$$

Problem 1.8 (1) Find a formula for the complex Fourier coefficients c_n in terms of the (real) Fourier coefficients a_n and b_n.
Hint: Use the identities $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.
(2) Find formulas for the (real) Fourier coefficients a_n and b_n, in terms of the complex Fourier coefficients c_n.
Hint: Use the previous item.

Problem 1.9 At which points x of the interval $[-L, L]$ does the Fourier series of f converge to $f(x)$?

$$\textbf{1.}\ f(x) = x\,, \qquad \textbf{2.}\ f(x) = x^2.$$

Problem 1.10 Solve the heat equation on a rod with zero temperature at finite ends ($0 \le x \le L, t \ge 0$):

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } 0 < x < L\,,\ t > 0\,, \\ u(0,t) \;=\; 0\,, \quad u(L,t) \;=\; 0\,, & \text{if } t > 0\,, \\ u(x,0) \;=\sin\frac{2\pi x}{L}\,, & \text{if } 0 < x < L\,. \end{cases}$$

Problem 1.11 Solve the heat equation on a rod with isolated endpoints ($0 \le x \le L, t \ge 0$):

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) \;=\; k\,\frac{\partial^2}{\partial x^2}u(x,t)\,, & \text{if } 0 < x < L\,,\ t > 0\,, \\ \frac{\partial}{\partial t}u(0,t) \;=\; 0\,, \quad \frac{\partial}{\partial t}u(L,t) \;=\; 0\,, & \text{if } t > 0\,, \\ u(x,0) \;=\; f(x)\,, & \text{if } 0 < x < L\,. \end{cases}$$

Problem 1.12 Solve the Laplace equation on the square $\{(x,y) \in \mathbb{R}^2\colon\ 0<x<\pi,\ 0<y<\pi\}$:

$$\begin{cases} \Delta u = \frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0\,, \\ u(0,y) = u(\pi,y) = 0\,, \\ u(x,0) = 2\sin 3x\,,\ u(x,\pi) = 0\,. \end{cases}$$

Hint: Check that the function $v(y) = c_1 e^{\mu y} + c_2 e^{-\mu y} = k_1 \cosh \mu(\pi - y) + k_2 \sinh \mu(\pi - y)$ is solution of the differential equation $v''(y) - \mu^2 v(y) = 0$.