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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.2: Fourier transform

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2 Fourier transform

Problem 2.1 Prove that if $f \in L^1(\mathbb{R})$ and $f > 0$, then $|\hat{f}(\omega)| < \hat{f}(0)$ for every $\omega \neq 0$.

Hint: The inequality $|\hat{f}(\omega)| \leq \hat{f}(0)$ is easy. If α denotes the complex argument of $\hat{f}(\omega)$, then $|\hat{f}(\omega)| = \hat{f}(\omega)\, e^{-i\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{i(\omega x - \alpha)} dx$. Now, take real parts in the equality $|\hat{f}(\omega)| = \hat{f}(0)$ to conclude that, a fortiori, $\omega = 0$.

Problem 2.2 Given $\alpha > 0$, compute the Fourier transform of the following functions, if we define the function $\chi_{[a,b]}(x)$ by

$$\chi_{[a,b]}(x) = \begin{cases} 1\,, & \text{if } x \in [a,b]\,, \\ 0\,, & \text{if } x \notin [a,b]\,. \end{cases}$$

1) $f(x) = e^{-\alpha|x|}$,
2) $f(x) = \frac{2\alpha}{x^2+\alpha^2}$,
3) $f(x) = \chi_{[-\alpha,\alpha]}(x)$,
4) $f(x) = x\chi_{[-\alpha,\alpha]}(x)$,
5) $f(x) = \chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)$,
6) $f(x) = |x|\chi_{[-\alpha,\alpha]}(x)$,
7) $f(x) = \frac{1}{x}$,
8) $f(x) = \frac{\sin \alpha x}{x}$,
9) $f(x) = (\alpha - |x|)\,\chi_{[-\alpha,\alpha]}$,
10) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
11) $f(x) = \frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}$,
12) $f(x) = \frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}$,

Problem 2.3 Let $f \in L^1(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Prove the following formulas:

(1) $\mathcal{F}\big[e^{i\alpha x} f(x)\big](\omega) = \mathcal{F}\big[f\big](\omega + \alpha)$.

(2) $\mathcal{F}\big[f(x-\alpha)\big](\omega) = e^{i\alpha\omega}\mathcal{F}\big[f\big](\omega)$.

(3) $\mathcal{F}\big[f(\alpha x)\big](\omega) = \frac{1}{|\alpha|}\mathcal{F}\big[f\big]\big(\frac{\omega}{\alpha}\big)$.

(4) $\mathcal{F}\big[\,\overline{f}\,\big](\omega) = \overline{\mathcal{F}\big[f\big](-\omega)}$.

(5) $\mathcal{F}\big[f\big](\omega) = \overline{\mathcal{F}\big[f\big](-\omega)}$, if f just take real values.

Problem 2.4 Compute the Fourier transform of the Gaussian function $f(x) = e^{-x^2}$.

Hint: Recall that $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$. Assume that $\omega > 0$ (the case $\omega < 0$ can be obtained from the case $\omega > 0$, by using the previous problem). Consider the integral of $f(z) = e^{-z^2}$ along the closed curve which is the union of the segment from $-R$ to R, the segment from $-R - i\omega/2$ to $R - i\omega/2$, and the two vertical segments joining $-R$ and $-R - i\omega/2$, and R and $R - i\omega/2$, which is 0 by Cauchy integral theorem. After that, take the limit as $R \to \infty$.

Problem 2.5 Compute the Fourier transform of the function $f(x) = e^{-ix^2}$.

Hint: Prove first that $\int_{\mathbb{R}} e^{-ix^2} dx = \sqrt{\pi}\, e^{-i\pi/4}$. In order to prove this formula, consider the integral of $f(z) = e^{-iz^2}$ along the closed curve which is the union of the segment from R to 0, the segment from 0 to $Re^{-i\pi/4}$, and the arc of the circumference of radius R from $Re^{-i\pi/4}$ to R, which is 0 by Cauchy integral theorem. After that, take the limit as $R \to \infty$.

Problem 2.6 Compute the Fourier transform of the function $f(x) = \sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}$.

Problem 2.7 For $\alpha > 0$, compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 \alpha x}{x^2}\, dx\,.$$

Hint: Use Plancherel's theorem and part 8) of Exercise 2.2.

Problem 2.8 Find a particular solution of the equation $u'' - u = f(x)$ by taking Fourier transforms in both sides of the equation.

Problem 2.9 Find a solution of the initial value problem for the heat equation in $\mathbb{R} \times (0, \infty)$ by taking Fourier transforms in the x-variable in both members of the equations:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 2.10 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t) + c \frac{\partial}{\partial x} u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 2.11 Find a solution of the initial value problem for the diffusion equation with convection:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) - 2 \frac{\partial}{\partial x} u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = e^{-x^2}, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 2.12 Find a solution of the initial value problem for the diffusion equation with absorption:

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t) - c\, u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t > 0\,, \\ u(x,0) = f(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 2.13 Find a solution of the initial value problem for the wave equation on $\mathbb{R} \times \mathbb{R}$

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t \in \mathbb{R}\,, \\ u(x,0) = f(x)\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t} u(x,0) = g(x)\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Problem 2.14 Prove that the function you have found in the previous exercise

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\; ds$$

is, in fact, a solution of the wave equation on $\mathbb{R} \times \mathbb{R}$ if f is of C^2-class (continuous with two continuous derivatives) on $\mathbb{R}$ and g is of C^1-class (continuous with one continuous derivative) on $\mathbb{R}$.

Problem 2.15 Find a solution of the initial value problem for the non-homogeneous wave equation in $\mathbb{R} \times \mathbb{R}$:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) + 6\,, & \text{if } x \in \mathbb{R}\,,\ t \in \mathbb{R}\,, \\ u(x,0) = x^2\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t} u(x,0) = 4x\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

Hint: Prove that if $u(x,t)$ is solution of this problem, then the function $v(x,t) = u(x,t) - 3t^2$ satisfies

$$\begin{cases} \frac{\partial^2}{\partial t^2} v(x,t) \;=\; \frac{\partial^2}{\partial x^2} v(x,t)\,, & \text{if } x \in \mathbb{R}\,,\ t \in \mathbb{R}\,, \\ v(x,0) \;=\; x^2\,, & \text{if } x \in \mathbb{R}\,, \\ \frac{\partial}{\partial t} v(x,0) \;=\; 4x\,, & \text{if } x \in \mathbb{R}\,. \end{cases}$$

FOURIER TRANSFORMS TABLE

$(x_0 \in \mathbb{R}, \alpha, \beta > 0)$

$$(TF1)\quad \mathcal{F}\big[e^{-\alpha x^2}\big](\omega) \;=\; \frac{1}{\sqrt{4\pi\alpha}}\, e^{-\omega^2/(4\alpha)},$$

$$(TF2)\quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-x^2/(4\alpha)}\Big](\omega) \;=\; e^{-\alpha\omega^2},$$

$$(TF3)\quad \mathcal{F}\big[e^{-\alpha|x|}\big](\omega) \;=\; \frac{\alpha}{\pi(\omega^2+\alpha^2)}\,,$$

$$(TF4)\quad \mathcal{F}\Big[\frac{2\alpha}{x^2+\alpha^2}\Big](\omega) \;=\; e^{-\alpha|\omega|},$$

$$(TF5)\quad \mathcal{F}\big[\chi_{[-\alpha,\alpha]}(x)\big](\omega) \;=\; \frac{\sin\alpha\omega}{\pi\omega}\,,$$

$$(TF6)\quad \mathcal{F}\Big[\frac{\sin\alpha x}{x}\Big](\omega) \;=\; \frac{1}{2}\,\chi_{[-\alpha,\alpha]}(\omega)\,,$$

$$(TF7)\quad \mathcal{F}\big[x\chi_{[-\alpha,\alpha]}(x)\big](\omega) \;=\; i\,\frac{\sin\alpha\omega - \alpha\omega\cos\alpha\omega}{\pi\omega^2}\,,$$

$$(TF8)\quad \mathcal{F}\big[\chi_{[0,\alpha]}(x) - \chi_{[-\alpha,0]}(x)\big](\omega) \;=\; i\,\frac{1-\cos\alpha\omega}{\pi\omega}\,,$$

$$(TF9)\quad \mathcal{F}\big[|x|\chi_{[-\alpha,\alpha]}(x)\big](\omega) \;=\; \frac{\alpha\omega\sin\alpha\omega + \cos\alpha\omega - 1}{\pi\omega^2}\,,$$

$$(TF10)\quad \mathcal{F}\big[(\alpha-|x|)\chi_{[-\alpha,\alpha]}(x)\big](\omega) \;=\; \frac{1-\cos\alpha\omega}{\pi\omega^2} \;=\; \frac{\sin^2(\alpha\omega/2)}{2\pi\omega^2}\,,$$

$$(TF11)\quad \mathcal{F}\big[e^{-i\alpha x^2}\big](\omega) \;=\; \frac{1}{\sqrt{4\pi\alpha}}\, e^{-i\pi/4}\, e^{i\omega^2/(4\alpha)},$$

$$(TF12)\quad \mathcal{F}\Big[\sqrt{\frac{\pi}{\alpha}}\, e^{-i\pi/4}\, e^{ix^2/(4\alpha)}\Big](\omega) \;=\; e^{-i\alpha\omega^2},$$

$$(TF13)\quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} + \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) \;=\; e^{-\alpha|\omega|}\cos x_0\omega\,,$$

$$(TF14)\quad \mathcal{F}\Big[\frac{\alpha}{(x-x_0)^2+\alpha^2} - \frac{\alpha}{(x+x_0)^2+\alpha^2}\Big](\omega) \;=\; ie^{-\alpha|\omega|}\sin x_0\omega\,,$$

$$(TF15)\quad \mathcal{F}\Big[\frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}\Big](\omega) \;=\; \frac{1}{2\alpha\beta(\alpha^2-\beta^2)}\big(\alpha e^{-\beta|\omega|} - \beta e^{-\alpha|\omega|}\big)\,,$$

$$(TF16)\quad \mathcal{F}\Big[\frac{1}{x}\Big](\omega) \;=\; \begin{cases} -i/2\,, & \text{if } \omega < 0\,, \\ 0\,, & \text{if } \omega = 0\,, \\ i/2\,, & \text{if } \omega > 0\,, \end{cases} \qquad \text{(it's understood as the principal value).}$$