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Departamento de Matemáticas

Complex variable and transforms. Problems

Chapter 2: Transforms

Section 2.3: Laplace transform

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3 Laplace transform

Problem 3.1 Given $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, calculate the Laplace's transform of the following functions:

1) $f(t) = 1\,,$	2) $f(t) = t^n\,,$
3) $f(t) = \cos at\,,$	4) $f(t) = \sin at\,,$
5) $f(t) = \frac{\sin at}{t}\,,$	6) $f(t) = e^{at}\,,$
7) $f(t) = t^n e^{at}\,,$	8) $f(t) = e^{at}\sin bt\,,$
9) $f(t) = e^{at}\cos bt\,,$	10) $f(t) = e^{at}\sinh bt\,,$
11) $f(t) = e^{at}\cosh bt\,,$	12) $f(t) = t\cos at\,,$
13) $f(t) = t^2 e^{-t}\cos t\,,$	14) $f(t) = t\int_0^t e^{-x}\sin x\,dx\,,$
15) $f(t) = \cos^3 t\,,$	16) $f(t) = \begin{cases} 0\,, & \text{if } 0 \le t \le 1\,, \\ (t-1)^k, & \text{if } t > 1\,, \end{cases} \quad (k \in \mathbb{N})\,.$

Problem 3.2 Calculate the inverse Laplace transform of the following functions:

$$1)\ F(z) = \frac{e^{3z}}{z^3+4z}\,, \qquad 2)\ F(z) = \frac{e^{-3z}}{z^3+4z}\,,$$

$$3)\ F(z) = \frac{1}{(z^2+a^2)^2}\,, \qquad 4)\ F(z) = \frac{z}{(z^2+a^2)^2}\,,$$

$$5)\ F(z) = \log(1+1/z)\,.$$

Problem 3.3 a) If $F(z)$ denotes the Laplace transform of f and $f \ge 0$ (or $f \le 0$), prove the identity

$$\int_0^\infty \frac{f(x)}{x}\,dx = \int_0^\infty F(s)\,ds\,.$$

b) Use this identity to compute

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,dx\,.$$

Problem 3.4 Calculate the integral $f(t) = \displaystyle\int_0^\infty \frac{\cos xt}{1+x^2}\,dx\,.$

Hint: Calculate $F(z) = \mathcal{L}[f](z)$ using Fubini's theorem. Then, calculate the inverse Laplace transform of $F(z)$.

Problem 3.5 Solve the following initial value problems, obtaining first the Laplace transform $Y(z)$ of the solution $y(t)$ and then, anti-transforming $Y(z)$:

$$1)\ \begin{cases} y' - 5y = \cos 3t\,, \\ y(0) = 1/2\,, \end{cases} \qquad 2)\ \begin{cases} y'' + 16y = \cos 4t\,, \\ y(0) = 0\,,\ y'(0) = 1\,, \end{cases}$$

$$3)\ \begin{cases} y'' - 6y' + 9y = t^2 e^{3t}\,, \\ y(0) = 2\,,\ y'(0) = 6\,, \end{cases} \qquad 4)\ \begin{cases} y'' + 4y' + 6y = 1 + e^{-t}\,, \\ y(0) = 0\,,\ y'(0) = 0\,. \end{cases}$$

Problem 3.6 Solve the initial value problem for the system of differential equations:

$$\begin{cases} x' - 6x + 3y = 8e^t\,, \\ y' - 2x - y = 4e^t\,, \\ x(0) = -1\,,\ y(0) = 0\,. \end{cases}$$

Problem 3.7 Solve the following initial value problems:

$$1)\ \begin{cases} y''+y'=\begin{cases} t+1 & \text{if } 0<t<1\,,\\ 3-t & \text{if } t>1\,,\end{cases} \\ y(0)=-1,\ y'(0)=0\,. \end{cases} \qquad 2)\ \begin{cases} y''+4y=\begin{cases} \cos 2t & \text{if } 0<t<2\pi\,,\\ 0 & \text{if } t>2\pi\,,\end{cases} \\ y(0)=y'(0)=0\,. \end{cases}$$

Problem 3.8 Let us consider the differential equation $tx''+2x'+tx=0$ for $t>0$.

a) Find the differential equation that verifies the Laplace transform $X(s)$ of $x(t)$ with the initial data $x(0)=1$.

b) Solve the differential equation for $X(s)$ using the property $\lim_{s\to\infty} X(s)=0$.

c) Calculate the Laplace anti-transform $x(t)$.

Problem 3.9 a) Let us consider the Volterra integral equation

$$y(t)+\int_0^t k(t-x)\,y(x)\,dx=f(t)\,,$$

where f and k are known functions and $y(t)$ is the unknown function. Calculate the Laplace transform of $y(t)$ in terms of the transforms of k and f.
b) As an application, solve the Volterra equation

$$y(t)-2\int_0^t \cos(t-x)\,y(x)\,dx=e^{2t}\,.$$

Problem 3.10 Solve, for $\omega\neq\omega_0$, the initial value problem

$$\begin{cases} x''+\omega_0^2\,x=k\sin\omega t\,, & \text{if } t>0\,,\\ x(0)=x'(0)=0\,, \end{cases}$$

which describes the forced oscillations of a mass in a not damped spring. What happens if $\omega=\omega_0$? Explain physically the obtained results.

Problem 3.11 Solve the initial value problem for the system of differential equations:

$$\begin{cases} x'=4x-y\,,\\ y'=2x+y\,,\\ x(0)=0\,,\ y(0)=1\,. \end{cases}$$

Problem 3.12 Solve the initial value problem:

$$\begin{cases} x'=5x+4y\,,\\ y'=-x+y\,,\\ x(0)=0\,,\ y(0)=1\,. \end{cases}$$

Problem 3.13 Solve the integral equation

$$f(t)=4t+\int_0^t f(t-s)\sin s\,ds\,.$$

Problem 3.14 Find a function satisfying $f(0) = 1$ and

$$f'(x) = 1 - \int_0^x f(x-s)\, e^{-2s}\, ds\,.$$

Problem 3.15 Find a function satisfying $f(0) = -1$ and

$$f'(x) + \int_0^x f(t)\, dt = x^2 - x + 2\,.$$

LAPLACE TRANSFORMS TABLE

$$f(t) = 1\,, \qquad \mathcal{L}[f](z) = \frac{1}{z} \qquad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = t^n \quad (n \in \mathbb{N})\,, \qquad \mathcal{L}[f](z) = \frac{n!}{z^{n+1}} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = t^a \quad (a > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(a+1)}{z^{a+1}} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2 + a^2} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2 + a^2} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = \frac{\sin at}{t} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \arctan \frac{a}{z} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = e^{at} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{1}{z-a} \quad (\mathrm{Re}\, z > a)\,,$$

$$f(t) = e^{at} t^b \quad (a \in \mathbb{R},\, b > -1)\,, \qquad \mathcal{L}[f](z) = \frac{\Gamma(b+1)}{(z-a)^{b+1}} \quad (\mathrm{Re}\, z > a)\,,$$

$$f(t) = e^{at} \sin bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{b}{(z-a)^2 + b^2} \quad (\mathrm{Re}\, z > a)\,,$$

$$f(t) = e^{at} \cos bt \quad (a, b \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z-a}{(z-a)^2 + b^2} \quad (\mathrm{Re}\, z > a)\,,$$

$$f(t) = \sinh at = \frac{e^{at} - e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{a}{z^2 - a^2} \quad (\mathrm{Re}\, z > |a|)\,,$$

$$f(t) = \cosh at = \frac{e^{at} + e^{-at}}{2} \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{z}{z^2 - a^2} \quad (\mathrm{Re}\, z > |a|)\,,$$

$$f(t) = \sin at - at \cos at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2a^3}{(z^2 + a^2)^2} \quad (\mathrm{Re}\, z > 0)\,,$$

$$f(t) = t \sin at \quad (a \in \mathbb{R})\,, \qquad \mathcal{L}[f](z) = \frac{2az}{(z^2 + a^2)^2} \quad (\mathrm{Re}\, z > 0)\,.$$